THE TRANSVERSE ANGULAR MOMENTUM SUM RULE

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We explain the origin of the controversy about the existence of a transverse angular momentum sum rule, and show that it stems from utilizing an incorrect result in the literature, concerning the expression for the expectation values of the angular momentum operators. We demonstrate a new, short and direct way of obtaining correct expressions for these expectation values, from which a perfectly good transverse angular momentum sum rule can be deduced. We also introduce a new classification of sum rules into *primary* and *secondary* types. In the former all terms occurring in the sum rule can be measured experimentally; in the latter some terms cannot be measured experimentally.

Keywords: nucleon, angular momentum, sum rule.

1. Background

Shore and White's claim[1] that a_0 (which in the simple parton model is equal to the contribution to the nucleon's angular momentum arising from the quark spins) does *not* contribute to the nucleon's angular momentum, surprised us. Their analysis was based on a classical paper of Jaffe and Manohar,[2] who stressed the subtleties and warned that 'a careful limiting procedure has to be introduced'. Trying to understand this we became convinced that despite all the care, there are flaws. With the J-M result one cannot have a sum rule for a transversely polarized nucleon. With the correct version[3] one can!

2. Why the problem is non-trivial

What is the aim? We consider a nucleon with 4-momentum p^μ and covariant spin vector S^μ corresponding to some specification of its spin state e.g. helicity, transversity or spin along the Z-axis i.e. a nucleon in state $|p, S\rangle$. We require an expression for the expectation value of the angular momentum

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in this state i.e. for $\langle p, S|\boldsymbol{J}|p, S\rangle$ i.e. we require an expression in terms of p and S. This can then be used to relate the expectation value of $\boldsymbol{J}$ for the nucleon to the angular momentum carried by its constituents.

2.1. *The traditional approach*

In every field theory there is an expression for the angular momentum density operator. The angular momentum operator $\boldsymbol{J}$ is then an integral over all space of this density. Typically the angular momentum density involves the energy-momentum tensor density $T^{\mu\nu}(x)$ in the form e.g.

$$\boldsymbol{J}_z = \boldsymbol{J}^3 = \int dV [x T^{02}(\boldsymbol{x}) - y T^{01}(\boldsymbol{x})]$$

Consider the piece $T^{02}(\boldsymbol{x})$. It is a local operator, so by translational invariance of the theory

$$T^{02}(\boldsymbol{x}) = e^{i\boldsymbol{P}.\boldsymbol{x}} T^{02}(0) e^{-i\boldsymbol{P}.\boldsymbol{x}}$$

where $\boldsymbol{P}$ are the linear momentum operators i.e. the generators of translations. Now the nucleon is in an eigenstate of momentum, so $\boldsymbol{P}$ acting on it just becomes $\boldsymbol{p}$. The numbers $e^{i\boldsymbol{p}.\boldsymbol{x}} e^{-i\boldsymbol{p}.\boldsymbol{x}}$ cancel out and we are left with:

$$\int dV x \, \langle p, S | T^{02}(0) | p, S \rangle$$

The matrix element is independent of x so we are faced with $\int dV x = \infty$? or $= 0$? Totally ambiguous! The problem is an old one: In ordinary QM plane wave states give infinities. The solution is an old one: Build a wave packet, a superposition of physical plane wave states. Now Jaffe and Manohar are generally very careful, but nonetheless there are errors in their derivation. They end up with the following expression for the matrix elements of the angular momentum operator:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s} | \boldsymbol{J}_i | \boldsymbol{p}, \boldsymbol{s} \rangle\rangle_{JM} = \frac{1}{4mp_0} \left[(3p_0^2 - m^2)\boldsymbol{s}_i - \frac{3p_0 + m}{p_0 + m} (\boldsymbol{p}.\boldsymbol{s})\boldsymbol{p}_i \right]$$

where $p^\mu = (p^0, \boldsymbol{p})$ and $\boldsymbol{s}_i$ are the components of the rest frame spin vector. Recall that the parton picture is supposed to be valid when the nucleon is viewed in a frame where it is moving very fast. In other words to derive a sum rule involving partons we must take the limit $p^0 \to \infty$. If we consider *longitudinal* spin i.e $\boldsymbol{p} \, / / \, \boldsymbol{s}$ one obtains:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s} | \boldsymbol{J}_i | \boldsymbol{p}, \boldsymbol{s} \rangle\rangle_{JM} = \frac{1}{2} \boldsymbol{s}_i \tag{1}$$

and there is no problem. But for transverse polarization one gets:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s}|\boldsymbol{J}_i|\boldsymbol{p}, \boldsymbol{s}\rangle\rangle_{JM} = \frac{1}{4mp_0}\left[(3p_0^2 - m^2)\boldsymbol{s}_i\right] \tag{2}$$

which $\to \infty$ as $p_0 \to \infty$, so no sum rule is possible. We will see in a moment that the result for transverse spin is incorrect. The J-M reaction to our criticism was very gracious and positive!

"Better late than never. Aneesh and I finally found ourselves in the same place with the time to review the issues you raised by email and in your recent paper. We agree that there is an error in our eq. (6.9). It came from treating the quantity u(p', s)u(p, s) with insufficient care. Thanks for taking care and finding this mistake. It's good to get it cleared up. I have to add that I found your paper rather difficult to read. There is quite a bit of stuff that gets in the way of the relatively simple error..........."

2.2. *A new approach*

It is simple. It is short. It works for any spin. Previous methods only work for spin 1/2. We know how rotations affect states. If $|\boldsymbol{p}, m\rangle$ is a state with momentum $\boldsymbol{p}$ and spin projection m in the rest frame of the particle, and if $\hat{R}_z(\beta)$ is the operator for a rotation β about OZ, then

$$\hat{R}_z(\beta)|\boldsymbol{p}, m\rangle = |\boldsymbol{R}_z(\beta)\boldsymbol{p}, m'\rangle\, D^s_{m'm}[R_z(\beta)] \tag{3}$$

where the $D^s_{m'm}$ are the standard rotation matrices for spin s. But rotations are generated by the angular momentum operators! i.e.

$$\hat{R}_i(\beta) = e^{-i\beta\boldsymbol{J}_i}$$

so that

$$\boldsymbol{J}_i = i\frac{d}{d\beta}\hat{R}_i(\beta)\big|_{\beta=0}$$

From Eq. (3) we know what the matrix element of $\hat{R}_i(\beta)$ looks like. So we simply differentiate and put $\beta = 0$. Thus we have

$$\langle\boldsymbol{p}', m'|\boldsymbol{J}_i|\boldsymbol{p}, m\rangle = i\frac{\partial}{\partial\beta}\langle\boldsymbol{p}', m'|R_i(\beta)|\boldsymbol{p}, m\rangle|_{\beta=0}$$

One technical point: you have to know that the derivative of the rotation matrix for spin s at $\beta = 0$ is just the spin matrix for that spin. e.g. for spin 1/2 just $\sigma_i/2$.

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3. Comparison of results

For the expectation values we find, for *any* spin configuration (longitudinal, transverse etc) the remarkably simple result (suppressing a delta-function term):

$$\langle\langle \boldsymbol{p}, \boldsymbol{s}|\boldsymbol{J}_i|\boldsymbol{p}, \boldsymbol{s}\rangle\rangle = \frac{1}{2}\boldsymbol{s}_i$$

This agrees precisely with the JM result for longitudinal spin Eq. (1). But for transverse polarization our result differs from the JM Eq. (2), which implied no possibility of a transverse sum rule. With our correct result there is no fundamental distinction between the transverse and longitudinal cases.

4. Sum rules

Consider a nucleon moving along OZ with momentum $\boldsymbol{p}$ and spin projection m along OZ. We expand the nucleon state as a superposition of n-parton Fock states.

$$|\boldsymbol{p}, m\rangle = \sum_n \sum_{\{\sigma\}} \int d^3\boldsymbol{k}_1 \dots d^3\boldsymbol{k}_n \, \psi_{\boldsymbol{p},m}(\boldsymbol{k}_1, \sigma_1, \dots \boldsymbol{k}_n, \sigma_n)$$
$$\times \, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 \dots - \boldsymbol{k}_n) \, |\boldsymbol{k}_1, \sigma_1, \dots \boldsymbol{k}_n, \sigma_n\rangle.$$

where σ_j labels the spin state of the parton, either a projection along OZ for quarks, or helicity for gluons.

There are two independent cases:

(a) *Longitudinal polarization* i.e. the nucleon rest frame spin vector $\boldsymbol{s}$ is along OZ. The sum rule for $\boldsymbol{J}_z$ yields the well known result

$$1/2 = 1/2\,\Delta\Sigma + \Delta G + \langle L_z^q\rangle + \langle L_z^G\rangle \tag{4}$$

(b) *Transverse polarization* i.e. $\boldsymbol{s} = \boldsymbol{s}_T$ where $\boldsymbol{s}_T \perp \boldsymbol{p}$. The sum rule for $\boldsymbol{J}_x$ or $\boldsymbol{J}_y$ yields a a *new* sum rule

$$1/2 = 1/2 \sum_{q,\,\bar{q}} \int dx\, \Delta_T q(x) + \sum_{q,\,\bar{q},\,G} \langle L_{\boldsymbol{s}_T}\rangle \tag{5}$$

Here $L_{\boldsymbol{s}_T}$ is the component of $\boldsymbol{L}$ along $\boldsymbol{s}_T$.

The structure functions $\Delta_T q^a(x) \equiv h_1^q(x)$ are known as the quark transversity or transverse spin distributions in the nucleon. As mentioned no such parton model sum rule is possible with the J-M formula for the

expectation value of $\boldsymbol{J}_i$ because for $i = x, y$ the matrix elements diverge as $p \to \infty$.

It is absolutely crucial to note that the sum rule Eq. (5) involves a *sum* of quark and antiquark densities. Not realizing this has led to some misunderstandings.

The *tensor charge* of the nucleon involves the *difference* of the first moments of quark and antiquark contributions. Thus the transverse spin sum rule, although it involves the transverse spin or transversity quark and antiquark densities, does *not* involve the nucleon's tensor charge. The tensor charge operator is *not* related to the angular momentum.

The structure functions $\Delta_T q(x) \equiv h_1^q(x)$ are most directly measured in doubly polarized Drell-Yan reactions

$$p(\boldsymbol{s}_T) + p(\boldsymbol{s}_T) \to l^+ + l^- + X$$

where the asymmetry is proportional to

$$\sum_f e_f^2 [\Delta_T q_f(x_1) \Delta_T \bar{q}_f(x_2) + (1 \leftrightarrow 2)].$$

They can also be determined from the asymmetry in semi-inclusive hadron-hadron interactions like

$$p + p(\boldsymbol{s}_T) \to H + X$$

where H is a detected hadron, typically a pion, and in semi-imclusive lepton-hadron reactions (SIDIS) with a transversely polarized target, like

$$\ell + p(\boldsymbol{s}_T) \to \ell + H + X.$$

The problem here is that in these semi-inclusive reactions $\Delta_T q(x)$ always occurs multiplied by the Collins fragmentation function, about which we are only at present gathering information.

5. A new classification of sum rules

Part of the reason that there are claims and counter-claims about the existence of certain sum rules is that different people have a different interpretation as to what a sum rule really implies. To clarify this we propose a new classification into *primary* and *secondary* sum rules.

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- A *primary* sum rule is one in which every term occurring in the sum rule can be measured experimentally. If the derivation of the sum rule is rigorous and if it fails experimentally, one can conclude that the theory behind it is incorrect. Examples are the Bjorken sum rule,[4] the ELT sum rule[5] and the Ji sum rule.[6]
- A *secondary* sum rule is one in which not every term occurring can be measured experimentally. Examples are Eq. (4) and Eq. (5), where we do not know how to measure the orbital angular momentum terms experimentally. Consequently a secondary sum rule can't test the validity of a theory, but this does not mean the sum rule is vacuous. It can tell us about the terms which we cannot measure, and that can be of value in model building or in understanding the structure of say the nucleon. Do not forget that the renaissance of spin dependent deep inelastic scattering, both theory and experiment, is a direct consequence of using a secondary sum rule i.e. Eq. (4) to proclaim the existence of a "spin crisis in the parton model".[7]

Of course the above is an idealization. I do not know of a single case where literally everything is measurable. So in the Bjorken and ELT sum rules one has to extrapolate $g_1(x)$ and $x[g_1(x) + 2g_2(x)]$ respectively to $x = 0$, and in the Ji case one must extrapolate $E(x, \xi, \Delta^2)$ to $\Delta^2 = 0$. Nonetheless I think the classification is useful.

6. Conclusions

In order to derive angular momentum sum rules we need an expression for the matrix elements of the angular momentum operators $\boldsymbol{J}$ in terms of the momentum $\boldsymbol{p}$ and spin $\boldsymbol{s}$ of the particle. Such matrix elements are divergent and ambiguous in the traditional approach. The infinities and ambiguities can be handled using wave packets, but the calculations are long and unwieldy. and the results, in some classic papers, are incorrect for a transversely polarized nucleon. Consequently it was claimed that no angular momentum rule was possible for a transversely polarized nucleon.

We have found a simple, direct method for evaluating these matrix elements, which is free of infinities and ambiguities. It uses the facts that we know how states transform under rotations, and that the rotation operators are exponentials of the generators of rotations i.e. of the angular momentum operators. It leads quickly and relatively painlessly to correct results.

The great success of the correct approach is that it allows the derivation of a sum rule also for transversely polarized nucleons.

Finally, we have proposed a classification of sum rules into *primary* and *secondary* sum rules, according to whether all, or not all, the terms in a sum rule can be measured experimentally. Whereas the former could, in principle, disprove a theory, the latter can only give us information about quantities which we cannot measure directly. *Both* the longitudinal and the transverse angular momentum sum rules are secondary.

Ackmowledgements

7. References

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