On the importance of Lorentz structure in the parton model: target mass corrections, transverse momentum dependence, positivity bounds

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We show that respecting the underlying Lorentz structure in the parton model has very strong consequences. Failure to insist on the correct Lorentz covariance is responsible for the existence of contradictory results in the literature for the polarized structure function $g_2(x)$, whereas with the correct imposition we are able to derive the Wandzura-Wilczek relation for $g_2(x)$ and the target-mass corrections for polarized deep inelastic scattering *without* recourse to the operator product expansion. We comment briefly on the problem of threshold behaviour in the presence of target-mass corrections. Careful attention to the Lorentz structure has also profound implications for the structure of the transverse momentum dependent parton densities often used in parton model treatments of hadron production, allowing the $\boldsymbol{k}_T$ dependence to be *derived explicitly*. It also leads to stronger positivity and Soffer-type bounds than usually utilized for the collinear densities.

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I. INTRODUCTION

In this paper we shall show that imposition of the correct Lorentz structure in the parton model allows us to relate two issues which, *a priori*, do not seem to be connected with each other: the derivation of a consistent expression for the polarized deep inelastic scattering (DIS) structure function $g_2(x)$ in the parton model, and the derivation of the higher twist target-mass corrections i.e. corrections of the form M^2/Q^2, where M is the nucleon mass, to $g_1(x,Q^2)$ and $g_2(x,Q^2)$.

As will be discussed below, the target-mass corrections have previously been derived in a very complicated way from the operator product expansion (OPE), and the derivation of the Wandzura-Wilczek (WW) expression for $g_2(x)$ has involved a questionable analytic continuation in the OPE moments [1]. It is thus particularly interesting that these results can be derived in a field theoretic context without use of the OPE. Target-mass corrections for *unpolarized* DIS were first derived by Nachtmann [2] employing a very elegant mathematical approach in which the power series expansion used in the OPE was replaced by an expansion into a series of hyperspherical functions (representation functions of the homogeneous Lorentz group). Later, also within the context of the OPE, Georgi and Politzer [3] re-derived Nachtmann's results using what they called an alternative analysis "for simple-minded souls like ourselves" i.e. based on a straightforward power series expansion but, in fact, requiring a very clever handling of the combinatoric aspects of the problem.

The derivation of target-mass corrections for *polarized* DIS turned out to be much more difficult. Several papers [4, 5] succeeded in expressing the reduced matrix elements a_n, d_n of the relevant operators in terms of combinations of moments of the structure functions, but did not manage to derive closed expressions for the structure functions $g_{1,2}$ themselves. The latter was finally achieved in 1997 by Piccione and Ridolfi [6] and later generalized to weak interaction, charged current reactions, by Blümlein and Tkabladze [7]. These calculations, based on the OPE, are extremely complicated, and we shall see presently how the same results can be obtained in a much simpler field-theoretic approach.

The clue to this entire approach is contained in the classic paper of Ellis, Furmanski and Petronzio (EFP) [8], which gave the first derivation of the dynamic higher twist corrections to unpolarized DIS in terms of amplitudes involving not just the "handbag" diagram of Fig. 1, whose soft part is the quark-quark (qq) correlator Φ, but the higher order diagrams in Fig. 2, whose soft parts are the qqG and $qqGG$ correlators respectively.

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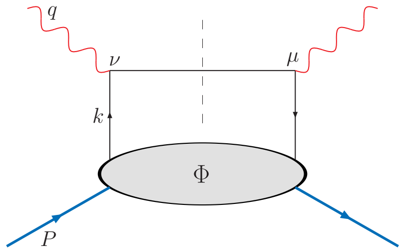

FIG. 1: The DIS "handbag" diagram involving the qq-correlator.

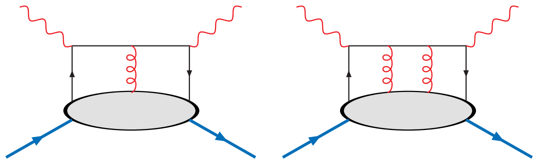

FIG. 2: DIS Diagrams involving the qqG and $qqGG$-correlators.

EFP begin with a brief discussion of a parton model, which they refer to as a "reference model", in which the active quark, momentum k^μ, emitted from the nucleon is on mass-shell. Handling the kinematics exactly they arrive at expressions for the unpolarized structure functions $F_{1,2}$ in terms of the quark densities $q(x)$ which are identical to those of Nachtmann [2].

At first sight it seems surprising that such a naive model should give the exact results of the theory. But the point is – and this is something not stressed in the literature – that any off-shellness of the quark is a direct consequence of QCD, i.e. if the strong interaction coupling $g = 0$ then it follows that (taking the quark mass $m_q = 0$) $k^2 = 0$ [see Eq. (5)]. Target-mass corrections are, by definition, kinematic in origin, therefore are independent of the value of g. Hence it is not miraculous that a model with $k^2 = 0$, i.e. equivalent to putting $g = 0$, should yield exact results for the target-mass corrections. However, it is crucial that the Lorentz structure is built into the model, as is done by EFP [8]. The implications of this are that the exact target-mass corrections must be derivable from the "handbag" diagram, Fig. 1, alone, since the diagrams of Fig. 2 vanish when $g = 0$.
We shall carry out the derivation for polarized DIS and also show that when the Lorentz structure is respected the $k^2 = 0$ "model" yields an unambiguous result for $g_2(x)$, namely the Wandzura-Wilczek (WW) result [1]

$$g_2(x_{\rm Bj}) = -g_1(x_{\rm Bj}) + \int_{x_{\rm Bj}}^1 {\rm d}y\, \frac{g_1(y)}{y}\,, \tag{1}$$

where $x_{\rm Bj}$ is the well-known Bjorken variable for DIS. This suggests that the analytic continuation necessary in the OPE derivation of the WW result is in fact correct, and also implies that in a correctly formulated parton model, with on-shell quarks, $g_2(x)$ is exactly given by the WW expression. For recent studies of the role of dynamical higher-twist contributions and possible violations of generalized WW relations see Refs. [9, 10].

In this paper we shall also show that careful attention to the Lorentz structure in the parton model imposes strong constraints on the possible $\boldsymbol{k}_T$ dependence of the so-called transverse momentum dependent (TMD) parton densities, that in recent years have received much emphasis (for an up-to-date review see e.g. Ref. [11] and references therein). Indeed, taking into account the transverse momentum of partons is important for an understanding of the large transverse single spin asymmetries observed in many reactions. It is also essential in order to generate the parton orbital angular momentum which appears necessary as a consequence of the small contribution to the nucleon angular momentum provided by the parton spins.

As we are going to show, imposing Lorentz covariance we are able to *derive*, for the unpolarized densities $q(x,\, \boldsymbol{k}_T^2)$, the longitudinal densities $\Delta q(x,\, \boldsymbol{k}_T^2)$ and the transversity densities $\Delta_T q(x,\, \boldsymbol{k}_T)$ or $\Delta'_T q(x,\, \boldsymbol{k}_T^2) \equiv h_1(x,\, \boldsymbol{k}_T^2)$, their dependence on $\boldsymbol{k}_T$ from the functional form of the usual collinear, $\boldsymbol{k}_T$-integrated, parton densities $q(x)$, $\Delta q(x)$ and $\Delta_T q(x)$ [These relations are spelled out in detail in Eqs. (57), (61) and (60), and in Eqs. (68), (70)]. Moreover, if perturbative evolution with Q^2 is taken into account, then the evolution of the $\boldsymbol{k}_T$-dependent densities is entirely controlled via the known evolution of the collinear densities.

It should be noted that the functional form found in the parton model is quite different from the often used factorized form, typically $f(x)\, e^{-\lambda\, \boldsymbol{k}_T^2}$, though the latter may be a reasonable starting point for analyzing the presently available data.

Further, we obtain positivity and Soffer-type bounds [12], based on the $\boldsymbol{k}_T$-dependent densities, which are stronger, in general, than those usually imposed on the purely x-dependent, $\boldsymbol{k}_T$-integrated, collinear densities, and we suggest that it would be interesting to impose these bounds on the collinear densities when extracting parton densities from deep inelastic and semi-inclusive deep inelastic scattering data.

Therefore, as will become even more clear in the following, careful attention to the Lorentz structure in the parton model with on-shell partons, i.e. with $k^2 = 0$, has dramatic consequences. Given the standard belief that the *soft* functions appearing in QCD have support only in a very narrow range of values of k^2 around $k^2 = 0$, it is tempting to suppose that the results derived in this paper may not be too different from those holding in a full QCD treatment.

Covariant parton models have been discussed in the literature for many decades, beginning with the work of Landshoff, Polkinghorne and Short in 1971 [13], and Franklin in 1977 [14]. These papers dealt only with the unpolarized structure functions $F_{1,2}(x)$. The collinear spin-dependent structure functions $g_{1,2}(x)$ were studied, in response to the "spin crisis in the parton model", by Jackson, Roberts and Ross [15], who also commented upon some aspects of quark transverse momentum, and by Franklin and Ierano [16, 17]. More recently, a very detailed study of covariant models was initiated by Zavada in 1997 [18], and later developed by Zavada [19, 20] and Efremov, Teryaev and Zavada [21], to include the polarized parton densities as well as parton transverse momentum. While our paper was in its last phase of preparation we became aware of a further paper by Efremov, Schweitzer, Teryaev and Zavada [22] (which we shall refer to as ESTZ) and a very latest work by Zavada [23] in which TMD parton distributions are discussed in detail.

In these papers the analysis is based upon reasonable assumptions about the structure of the quark wave function in the nucleon *rest* frame. Because the treatment is covariant the results can then be translated to any other Lorentz frame. It is remarkable that many of the results of this kind of analysis are in complete agreement with our more general, frame independent, treatment. The principal differences are that: *i*) in our treatment the three leading twist densities, the unpolarized density $q(x,\, \boldsymbol{k}_T^2)$, the longitudinal density $\Delta q(x,\, \boldsymbol{k}_T^2)$ and the transversity density $\Delta_T q(x,\, \boldsymbol{k}_T)$ or $\Delta_T' q(x,\, \boldsymbol{k}_T^2) \equiv h_1(x,\, \boldsymbol{k}_T^2)$ are *independent*, whereas in the Zavada et al. papers only two of these are independent; *ii*) we take into account target mass corrections. Because our approaches are so different, we believe it will be instructive to present our derivation in full.

The plan of the paper is the following: in section II we discuss the hadronic correlator for polarized DIS in the parton model, with $g = 0$, and give the general expression of the structure functions $g_1(x)$ and $g_2(x)$; in section III we evaluate explicitly $g_1(x)$ and $g_2(x)$ including target-mass corrections and derive the WW relation; in section IV we extend our analysis to the TMD quark distributions and discuss the consequences of imposing Lorentz invariance; in section V we examine the positivity and Soffer bounds for the $\boldsymbol{k}_T$-dependent unpolarized, longitudinally and transversely polarized quark distributions, discuss how their correct implementation leads to new, more stringent, bounds on the $\boldsymbol{k}_T$-integrated collinear distributions and show some phenomenological implications of this approach; finally, in section VI we give some additional comments and conclusions.

II. THE HADRONIC CORRELATOR Φ FOR $g = 0$

The correlator $\Phi_{ij}(P, S; k)$, where P^μ and S^μ are the momentum and spin-polarization 4-vectors for the nucleon ($S^2 = -1$) , k^μ is the quark 4-momentum and i, j are Dirac indices, is defined by

$$\Phi_{ij}(P, S; k) = \int \frac{\mathrm{d}^4 z}{(2\pi)^4}\, e^{ik\cdot z}\, \langle P, S|\bar{\psi}_j(0)\, \psi_i(z)|P, S\rangle \,. \tag{2}$$

To preserve colour gauge invariance a path-ordered Wilson line (or gauge-link) should be inserted between the quark fields. Since we are mostly interested in the $g = 0$ case, we will take this operator as the identity.

We ignore for the moment flavour and the quark charge – they are trivially reinstated at the end – and, as already stated, work with $m_q = 0$. By partial integration we see that, when $g = 0$,

$$\mathrm{Tr}[\Phi\, \not{k}] \;=\; i \int \frac{\mathrm{d}^4 z}{(2\pi)^4}\, e^{ik\cdot z}\, \langle P, S|\bar{\psi}(0)\, \not{\partial}\, \psi(z)|P, S\rangle \;=\; 0\,, \tag{3}$$

and that for any of the 16 Dirac matrices Γ ($= I,\, \gamma^\mu,\, \sigma^{\mu\nu},\, \gamma^\mu\gamma_5,\, i\gamma_5$),

$$\mathrm{Tr}[\Phi\, \Gamma\, \not{k}] = 0\,. \tag{4}$$

Finally, again by partial integration, we get for all k^2

$$k^2\, \Phi_{ij} = -\int \frac{\mathrm{d}^4 z}{(2\pi)^4}\, e^{ik\cdot z}\, \langle P, S|\bar{\psi}_j(0)\, \not{\partial}^2\, \psi_i(z)|P, S\rangle = 0\,, \tag{5}$$

implying that

$$\Phi \propto \delta(k^2)\,. \tag{6}$$

Φ is a 4×4 matrix in Dirac spin space. The first attempt to write down its most general form was made by Ralston and Soper [24] who, however, missed one term [25]. The full dynamical twist-two structure, including the three so-called "naively T-odd" amplitudes, contains twelve terms [26]. Eight of these, including all the "naively T-odd" amplitudes, are eliminated by the requirements of Eqs. (3)-(5) and two become related to each other. We are left with

$$\Phi\,(P,S;k) \overset{g=0}{=} A_3\, \not{k} + \frac{A_8}{M}\,(k\cdot S)\,\not{k}\,\gamma_5 + \frac{A_{11}}{M^2}\, i\,\sigma^{\mu\nu}\,\gamma_5\, k_\mu\,[(k\cdot S)\,P_\nu - (k\cdot P)\,S_\nu]\,. \tag{7}$$

The scalar functions $A_{3,8,11}$ are, in general, functions of $P\cdot k$ and k^2. In our case, bearing in mind Eq. (6), we shall later use

$$A_3 = \frac{1}{\pi M^2}\,\varphi_3\big(\frac{2P\cdot k}{M^2}\big)\,\delta(k^2)\,\theta[(P-k)^2] \tag{8}$$

and

$$A_{8,11} = -\,\frac{2}{\pi M^2}\,\varphi_{8,11}\big(\frac{2P\cdot k}{M^2}\big)\,\delta(k^2)\,\theta[(P-k)^2]\,, \tag{9}$$

where the factors are for later convenience and, following Ref. [8], the θ-function ensures that the nucleon remnant has positive energy. Notice again the factor $\delta(k^2)$ in these expressions.

For the moment we shall only deal with the part relevant to $g_{1,2}(x_{\rm Bj})$ i.e. A_8. Polarized DIS is controlled by the antisymmetric part $W_A^{\mu\nu}$ of the hadronic tensor, related to the structure functions $g_{1,2}(x_{\rm Bj})$ via [27]

$$W_A^{\mu\nu} = \frac{2M}{P\cdot q}\,\epsilon^{\mu\nu\alpha\beta}\,q_\alpha\,\Big\{S_\beta\,g_1 + \Big[\,S_\beta - \frac{(S\cdot q)\,P_\beta}{P\cdot q}\,\Big]\,g_2\Big\}\,, \tag{10}$$

where q is the photon 4-momentum and our convention is $\epsilon_{0123}=1$.

The contribution from the handbag diagram, Fig. 1, to $W_A^{\mu\nu}$ is then

$$\begin{aligned} W_A^{\mu\nu} &= \epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}^4k\,(k_\rho+q_\rho)\,\delta[(k+q)^2]\,\mathrm{Tr}(\gamma_\sigma\,\gamma_5\,\Phi) \\ &= -\,\epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}^4k\,(k_\rho+q_\rho)\,\delta[(k+q)^2]\,\mathrm{Tr}(\gamma_\sigma\,\not{k})\,\frac{A_8}{M}\,(k\cdot S) \\ &= -\,\frac{4}{M}\,\epsilon^{\mu\nu\rho\sigma} q_\rho\,S^\beta I_{\beta\sigma}\,, \end{aligned} \tag{11}$$

where

$$I_{\beta\sigma} = \int \mathrm{d}^4k\,k_\beta\,k_\sigma\,\delta[(k+q)^2]\,A_8\,. \tag{12}$$

Note that the hadronic tensor $W_A^{\mu\nu}$ in Eq. (11) is electromagnetic gauge invariant.

We introduce the standard two auxiliary null vectors p^μ and n^μ , $p^2=n^2=0$, $p\cdot n=1$, in a slightly unusual way via

$$p^\mu = \frac{1}{D}\,(P^\mu - \frac{M^2\xi}{Q^2}\,q^\mu) \qquad\qquad n^\mu = \frac{2\xi}{DQ^2}\,(q^\mu + \xi\,P^\mu)\,, \tag{13}$$

where ξ is the Nachtmann variable

$$\xi = \frac{2x_{\rm Bj}}{1+\sqrt{1+4M^2x_{\rm Bj}^2/Q^2}}\,, \tag{14}$$

and

$$D \equiv 1+\epsilon \qquad\qquad \epsilon \equiv \frac{M^2\xi^2}{Q^2}\,. \tag{15}$$

Note that

$$q \cdot p = \frac{Q^2}{2\xi} \qquad q \cdot n = -\xi \qquad x_{\mathrm{Bj}} = \frac{\xi}{N} \qquad N \equiv 1 - \epsilon \, . \tag{16}$$

Switching off the target-mass corrections thus corresponds to putting $\epsilon = 0$, $D = N = 1$.

Since by definition, see Eq. (12), $I_{\beta\sigma}$ is symmetric under $\beta \longleftrightarrow \sigma$, its most general expression can be written as

$$I_{\beta\sigma} = M^2 B_g \, g_{\beta\sigma} + M^2 B_{pn} \, (\, p_\beta n_\sigma + p_\sigma n_\beta \,) + B_p \, p_\beta p_\sigma + M^4 B_n \, n_\beta n_\sigma \, . \tag{17}$$

On the other hand, from Eq. (12), bearing in mind the factor $\delta(k^2)$ in A_8, Eq. (9), and introducing the shorthand

$$\langle \, X \, \rangle \equiv \int \mathrm{d}^4 k \, X \, \delta[(k+q)^2] \, A_8 \, , \tag{18}$$

one finds that

$$\begin{aligned} g^{\beta\sigma} \, I_{\beta\sigma} &= \langle \, k^2 \, \rangle = 0 \\ p^\beta n^\sigma \, I_{\beta\sigma} &= \langle \, (k \cdot p) \, (k \cdot n) \, \rangle \; \equiv \; M^2 C_{pn} \\ p^\beta p^\sigma \, I_{\beta\sigma} &= \langle \, (k \cdot p)^2 \, \rangle \; \equiv \; M^4 C_p \\ n^\beta n^\sigma \, I_{\beta\sigma} &= \langle \, (k \cdot n)^2 \, \rangle \; \equiv \; C_n \, . \end{aligned} \tag{19}$$

By comparing with Eq. (17) and expressing the factors B_i in terms of the C_i's one can then write

$$I_{\beta\sigma} = M^2 C_{pn} \left[\, 2(p_\beta n_\sigma + p_\sigma n_\beta) - g_{\beta\sigma} \, \right] + C_n \, p_\beta p_\sigma + M^4 C_p \, n_\beta n_\sigma \, . \tag{20}$$

Moreover, from Eq. (13), because of the factor $\epsilon^{\mu\nu\rho\sigma} q_\rho \, S^\beta$ in Eq. (11) and the fact that $P \cdot S = 0$, we can replace

$$p_\sigma \to \tfrac{1}{D} P_\sigma \qquad n_\sigma \to \tfrac{2\xi^2}{DQ^2} P_\sigma \qquad p_\beta \to -\tfrac{M^2 \xi}{DQ^2} q_\beta \qquad n_\beta \to \tfrac{2\xi}{DQ^2} q_\beta$$

in Eq. (20). Thus

$$I_{\beta\sigma} \to -M^2 \, C_{pn} \, g_{\beta\sigma} + \frac{\xi M^2}{D^2 Q^2} \left[\, 4N C_{pn} - C_n + 4 \, \epsilon \, C_p \, \right] q_\beta P_\sigma \, . \tag{21}$$

Putting this into Eq. (11) and comparing with Eq. (10), we can read off directly expressions for $g_1 + g_2$ and g_2 and thereby obtain

$$g_1(x_{\mathrm{Bj}}) + g_2(x_{\mathrm{Bj}}) = 2 \, (P \cdot q) \, C_{pn} \, , \tag{22}$$

and

$$g_1(x_{\mathrm{Bj}}) = 2 \, (P \cdot q) \left\{ \, C_{pn} - \frac{N}{2D^2} \left[\, 4N \, C_{pn} - C_n + 4 \, \epsilon \, C_p \, \right] \right\} , \tag{23}$$

where we have used Eq. (16).

III. TARGET-MASS CORRECTIONS AND THE WANDZURA-WILCZEK RELATION

In order to evaluate the coefficients C_{pn}, C_p, C_n in Eqs. (22), (23) we parameterize k^μ in the usual way, bearing in mind that $k^2 = 0$,

$$k^\mu = x \, p^\mu + \frac{\boldsymbol{k}_T^2}{2x} \, n^\mu + k_T^\mu \, , \tag{24}$$

where $k_T^2 = -\boldsymbol{k}_T^2$. Then

$$\int \mathrm{d}^4 k = \frac{1}{2} \int \frac{\mathrm{d}x}{x} \, \mathrm{d}^2 \boldsymbol{k}_T \, \mathrm{d}k^2 = \frac{\pi}{2} \int \frac{\mathrm{d}x}{x} \, \mathrm{d}\boldsymbol{k}_T^2 \, \mathrm{d}k^2 \, , \tag{25}$$

and in terms of these variables

$$C_{pn} = \langle \frac{\boldsymbol{k}_T^2}{2M^2} \rangle \qquad C_n = \langle x^2 \rangle \qquad C_p = \langle \Big[\frac{\boldsymbol{k}_T^2}{2xM^2}\Big]^2 \rangle \, . \tag{26}$$

Bearing in mind Eqs. (16) and (24) one has

$$\delta[(k+q)^2] = \frac{x}{\xi}\, \delta\big[\boldsymbol{k}_T^2 - Q^2\, \frac{x}{\xi}(\frac{x}{\xi} - 1)\big] \, . \tag{27}$$

We therefore integrate over $\boldsymbol{k}_T^2$, introduce

$$\eta \equiv \frac{2P\cdot k}{M^2} = x + \frac{\boldsymbol{k}_T^2}{xM^2} = \frac{Q^2}{M^2\xi^2}\,[\,D\,x - \xi\,] \, , \tag{28}$$

via Eq. (27), and have

$$\int \mathrm{d}x = \frac{\epsilon}{D} \int_\xi^1 \mathrm{d}\eta \, , \tag{29}$$

where the upper integration limit comes from the θ-function in Eqs. (8), (9), and the lower limit from $\eta \geq x \geq \xi$. In terms of η the variables appearing in Eq. (26) are given by

$$x = \frac{1}{D}(\xi + \epsilon\eta) \qquad\qquad \frac{\boldsymbol{k}_T^2}{xM^2} = \frac{1}{D}(\eta - \xi) \, . \tag{30}$$

We substitute the expressions for the coefficients, Eq. (26), into Eqs. (22), (23) using Eq. (9), restore the quark charge, separate the leading twist terms, i.e. those that do not vanish when $\epsilon \to 0$, and, inside the integral, order the rest according to the powers of ϵ and η involved, to obtain, for each flavour,

$$g_1(x_{\rm Bj}) = e_q^2\, \frac{N}{2D^5} \int_\xi^1 \mathrm{d}\eta \,\Big\{\, \xi(-2\xi + \eta) + \epsilon\,(8\xi^2 - 11\xi\eta + 2\eta^2) + \epsilon^2(-2\xi^2 + 11\xi\eta - 8\eta^2) + \epsilon^3(-\xi\eta + 2\eta^2)\Big\}\, \varphi_8(\eta) \, , \tag{31}$$

$$g_1(x_{\rm Bj}) + g_2(x_{\rm Bj}) = e_q^2\, \frac{N}{2D^3} \int_\xi^1 \mathrm{d}\eta \,\Big\{\, \xi\,(\xi - \eta) + \epsilon\,\eta\,(\xi - \eta)\Big\}\, \varphi_8(\eta) \, . \tag{32}$$

From here we also get a separate parton-model expression for g_2:

$$g_2(x_{\rm Bj}) = e_q^2\, \frac{N}{2D^5} \int_\xi^1 \mathrm{d}\eta \,\Big\{\, \xi\,(3\xi - 2\eta) + \epsilon\,(-6\xi^2 + 10\xi\eta - 3\eta^2) + \epsilon^2(3\xi^2 - 10\xi\eta + 6\eta^2) + \epsilon^3(2\xi\eta - 3\eta^2)\Big\}\, \varphi_8(\eta) \, . \tag{33}$$

A. Neglecting target-mass corrections

The expression for $g_1(x_{\rm Bj})$, Eq. (31), must agree with the standard collinear parton model result, with no target-mass corrections (NTM), in terms of the (longitudinal) polarized quark density,

$$g_1^{\rm NTM}(x_{\rm Bj}) = \frac{e_q^2}{2}\, \Delta q(x_{\rm Bj}) \, , \tag{34}$$

when target masses are neglected, i.e. when $\epsilon \to 0$ and $\xi \to x_{\rm Bj}$. Thus

$$\Delta q(x_{\rm Bj}) = x_{\rm Bj} \int_{x_{\rm Bj}}^1 \mathrm{d}\eta\, (\eta - 2\,x_{\rm Bj})\, \varphi_8(\eta) \, . \tag{35}$$

This relation between $\varphi_8(\eta)$ and $\Delta q(x_{\rm Bj})$ allows us to express the integrals in Eq. (31) in terms of integrals over $\Delta q(\xi)$, but also has powerful consequences for the structure of transverse momentum dependent parton densities, as will be discussed shortly.

Turning to $g_1(x_{\rm Bj})+g_2(x_{\rm Bj})$, we have from Eq. (32):

$$g_1^{\rm NTM}(x_{\rm Bj})+g_2^{\rm NTM}(x_{\rm Bj})=\frac{e_q^2}{2}\,x_{\rm Bj}\int_{x_{\rm Bj}}^1 {\rm d}\eta\,(x_{\rm Bj}-\eta)\,\varphi_8(\eta)\,. \tag{36}$$

Consider now [see Eq. (35)]

$$\int_\xi^1 \frac{d\eta}{\eta}\,\Delta q(\eta)=\int_\xi^1 d\eta\int_\eta^1 d\eta'(\eta'-2\,\eta)\varphi_8(\eta')\,. \tag{37}$$

Changing the order of integration, we obtain ($\xi\le\eta\le\eta'\le 1$)

$$\int_\xi^1 \frac{{\rm d}\eta}{\eta}\,\Delta q(\eta)=\xi\int_\xi^1 {\rm d}\eta'(\xi-\eta')\,\varphi_8(\eta')\,. \tag{38}$$

Putting this relation in Eq. (36) and using Eq. (34) yields

$$g_1^{\rm NTM}(x_{\rm Bj})+g_2^{\rm NTM}(x_{\rm Bj})=\int_{x_{\rm Bj}}^1 \frac{{\rm d}x'}{x'}\,g_1^{\rm NTM}(x')\,, \tag{39}$$

which is the Wandzura-Wilczek relation in the absence of target-mass corrections.

B. Inclusion of target-mass corrections

Returning to Eqs. (31), (32) we note first that from Eqs. (38), (35)

$$\xi\int_\xi^1 {\rm d}\eta\,\varphi_8(\eta)=-\frac{1}{\xi}\left\{\Delta q(\xi)+\int_\xi^1\frac{{\rm d}\eta}{\eta}\,\Delta q(\eta)\right\}. \tag{40}$$

Then, using again Eq. (35) in the second step,

$$\begin{aligned}\int_\xi^1 {\rm d}\eta\,\eta\,\varphi_8(\eta) &= \int_\xi^1 {\rm d}\eta\,(\,\eta-2\xi+2\xi\,)\varphi_8(\eta)\\ &= \frac{\Delta q(\xi)}{\xi}+2\,\xi\int_\xi^1 {\rm d}\eta\,\varphi_8(\eta)\\ &= -\frac{1}{\xi}\left\{\Delta q(\xi)+2\int_\xi^1\frac{{\rm d}\eta}{\eta}\,\Delta q(\eta)\right\}.\end{aligned} \tag{41}$$

Further, integrating by parts,

$$\begin{aligned}\int_\xi^1 {\rm d}\eta\,\eta^2\,\varphi_8(\eta) &= \xi\int_\xi^1 {\rm d}\eta\,\eta\,\varphi_8(\eta)+\int_\xi^1{\rm d}\eta\int_\eta^1{\rm d}\eta'\,\eta'\,\varphi_8(\eta')\\ &= -\Delta q(\xi)-3\int_\xi^1\frac{{\rm d}\eta}{\eta}\,\Delta q(\eta)-2\int_\xi^1\frac{{\rm d}\eta}{\eta}\int_\eta^1\frac{{\rm d}\eta'}{\eta'}\,\Delta q(\eta')\,.\end{aligned} \tag{42}$$

Substituting Eqs. (40)-(42) into Eqs. (31), (32) we obtain expressions which, for reasons to be explained in the next subsection, we shall call the *bare parton model* (BPM) results

$$g_1^{\rm BPM}(x_{\rm Bj})=\frac{e_q^2}{2}\left\{\frac{N^2}{D^3}\,\Delta q(\xi)+\frac{4\epsilon N(1+N)}{D^4}\int_\xi^1\frac{{\rm d}\xi'}{\xi'}\,\Delta q(\xi')-\frac{4\epsilon N(N^2-2\epsilon)}{D^5}\int_\xi^1\frac{{\rm d}\xi'}{\xi'}\int_{\xi'}^1\frac{{\rm d}\xi''}{\xi''}\,\Delta q(\xi'')\right\} \tag{43}$$

$$g_1^{\rm BPM}(x_{\rm Bj})+g_2^{\rm BPM}(x_{\rm Bj})=\frac{e_q^2}{2}\left\{\frac{N}{D^2}\int_\xi^1\frac{{\rm d}\xi'}{\xi'}\,\Delta q(\xi')+\frac{2\epsilon N}{D^3}\int_\xi^1\frac{{\rm d}\xi'}{\xi'}\int_{\xi'}^1\frac{{\rm d}\xi''}{\xi''}\,\Delta q(\xi'')\right\} \tag{44}$$

$$g_2^{\rm BPM}(x_{\rm Bj}) = \frac{e_q^2}{2}\Big\{ -\frac{N^2}{D^3}\,\Delta q(\xi) + \frac{N^2(N-4\epsilon)}{D^4}\int_\xi^1 \frac{{\rm d}\xi'}{\xi'}\,\Delta q(\xi') + \frac{6\epsilon N^3}{D^5}\int_\xi^1 \frac{{\rm d}\xi'}{\xi'}\int_{\xi'}^1 \frac{{\rm d}\xi''}{\xi''}\,\Delta q(\xi'')\Big\}\,, \tag{45}$$

which, as can be shown after some algebra, agree exactly with the dynamical twist-two (i.e. taking $d_n = 0$) OPE results of Ref. [6]. Furthermore we find that

$$\xi\,\frac{\rm d}{{\rm d}\xi}\,[g_1^{\rm BPM}(x_{\rm Bj}) + g_2^{\rm BPM}(x_{\rm Bj})] = -\frac{D}{N}\,g_1^{\rm BPM}(x_{\rm Bj})\,, \tag{46}$$

so that, bearing in mind Eqs. (15) and (16),

$$g_1^{\rm BPM}(x_{\rm Bj}) + g_2^{\rm BPM}(x_{\rm Bj}) = \int_{x_{\rm Bj}}^1 \frac{{\rm d}x'}{x'}\,g_1^{\rm BPM}(x')\,, \tag{47}$$

and the Wandzura-Wilczek relation holds also when target-mass corrections are included [6].

C. Target mass corrections and threshold behaviour

There is a long-standing problem about the behaviour of the structure functions, both unpolarized and polarized, at $x_{\rm Bj} = 1$. To see this most simply consider the first term in the expansion of Eq. (43) in powers of M^2/Q^2, where we show the kinematic dependence in explicit detail

$$g_1(x_{\rm Bj}) = \frac{e_q^2}{2}\Delta q[\,x = \xi(x_{\rm Bj}, Q^2)\,]\,. \tag{48}$$

It is usually stated that one must have $g_1(x_{\rm Bj} = 1) = 0$, implying the bizarre result that $\Delta q(x)$ vanishes at $x = \xi(1, Q^2)$ i.e. at a continuous infinity of Q^2 dependent points. Of course without target mass corrections the vanishing of $g_1(x_{\rm Bj} = 1)$ simply implies that $\Delta q(x = 1) = 0$.

There are no satisfactory prescriptions for avoiding this issue in the literature. Georgi and Politzer [3] and Piccione and Ridolfi [6] argue that higher twist terms must be taken into account in the region of large $x_{\rm Bj}$, whereas Accardi and Melnitchouk [28] impose some constraints on the virtuality of the struck quark.

We would like to propose a very simple resolution to this problem. In the standard parton model treatment there is the underlying assumption that the struck quark and target fragments materialize into physical hadrons with probability one. This is tantamount to the assumption of *equivalent completeness* of partonic and hadronic states i.e.

$$\sum_{\substack{\text{all parton}\\ \text{states}}} |parton\rangle\langle parton| \;=\; \sum_{\substack{\text{all hadron}\\ \text{states}}} |hadron\rangle\langle hadron|\,. \tag{49}$$

While this might be reasonable for massless hadrons, it cannot be true when hadrons possess their physical masses. Thus what is needed is a *non-perturbative* function to express the failure of equivalent completeness. It has to express the fact that there is zero probability for the final partonic state (the struck parton and the target fragments) to hadronize if its total energy is too small to produce an inelastic hadronic event, and that the probability for hadronization is one when the energy is large enough. The simplest possibility is $\theta(x_{\rm Th} - x_{\rm Bj})$, where $x_{\rm Th}$ is the maximum kinematically allowed value of $x_{\rm Bj}$ for *inelastic* scattering. We note that, strictly speaking, and excluding photon production which is higher order in the fine structure constant, this is *not* $x_{\rm Th} = 1$, but

$$x_{\rm Th} = \frac{Q^2}{Q^2 + \mu(2M + \mu)}\,, \tag{50}$$

where μ is the pion mass.

Thus we propose that in a *physical parton model* (PPM) the BPM target mass dependent expressions, Eqs. (43), (44), for $g_1^{\rm BPM}$ and $g_1^{\rm BPM} + g_2^{\rm BPM}$ should be modified to

$$g_1^{\rm PPM}(x_{\rm Bj}) = g_1^{\rm BPM}(x_{\rm Bj})\,\theta(x_{\rm Th} - x_{\rm Bj}) \tag{51}$$

$$g_1^{\rm PPM}(x_{\rm Bj}) + g_2^{\rm PPM}(x_{\rm Bj}) = [g_1^{\rm BPM}(x_{\rm Bj}) + g_2^{\rm BPM}(x_{\rm Bj})]\,\theta(x_{\rm Th} - x_{\rm Bj})\,. \tag{52}$$

IV. RESULTS FOR THE TRANSVERSE MOMENTUM DEPENDENT QUARK DENSITIES

In the standard collinear parton model the transverse momentum $\boldsymbol{k}_T$ is integrated over up to the large scale Q. But in the last few years many polarized reactions have been studied in a more general framework in which the fundamental parton densities $q(x)$, $\Delta q(x)$ and $\Delta_T q(x)$ are allowed to depend on the *intrinsic* $\boldsymbol{k}_T$ (see e.g. Refs. [11, 29] and references therein). In all these approaches it is often assumed that the x and $\boldsymbol{k}_T$ dependence can be factorized. This, as we shall see, is in contradiction with the Lorentz structure of the amplitudes, *at least in the* $g = 0$ *case.* We shall comment later on the more general situation.

For unpolarized DIS the analogue of Eq. (35) is [8]

$$q(x_{\rm Bj}) = x_{\rm Bj} \int_{x_{\rm Bj}}^1 {\rm d}\eta \; \varphi_3(\eta) \, . \tag{53}$$

Moreover, on changing integration variables from η to $\boldsymbol{k}_T^2$ via Eq. (28), we see that $q(x, \boldsymbol{k}_T^2)$ for which

$$\int {\rm d}^2\boldsymbol{k}_T \, q(x, \boldsymbol{k}_T^2) = q(x) \tag{54}$$

must be given by

$$q(x, \boldsymbol{k}_T^2) = \frac{1}{\pi M^2} \, \varphi_3\big(x + \frac{\boldsymbol{k}_T^2}{xM^2}\big) \, \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2] \, . \tag{55}$$

Thus the x and $\boldsymbol{k}_T^2$ dependence are intimately linked. Moreover, the maximum size of $|\boldsymbol{k}_T|$ is bounded and x-dependent, and in general $\boldsymbol{k}_T^2 \le M^2/4 \simeq 0.25$ GeV^2.

Most importantly, if the functional form of $q(x_{\rm Bj})$ is known (for consistency it must, as usual, satisfy $q(1) = 0$) then from Eq. (53)

$$\varphi_3(\eta) = -\frac{\rm d}{{\rm d}\eta} \left[\frac{q(\eta)}{\eta}\right] . \tag{56}$$

Therefore we have the remarkable result that, in the $g = 0$ case, the $\boldsymbol{k}_T^2$ dependence of $q(x, \boldsymbol{k}_T^2)$ is completely determined by the $x_{\rm Bj}$ dependence of the standard collinear quark density $q(x_{\rm Bj})$:

$$q(x, \boldsymbol{k}_T^2) = -\frac{1}{\pi M^2} \frac{\rm d}{{\rm d}x} \left[\frac{q(x)}{x}\right]_{x=\eta} \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2] \, , \tag{57}$$

where, we remind the reader,

$$\eta = x + \frac{\boldsymbol{k}_T^2}{xM^2} \, . \tag{58}$$

Notice that Eqs. (56)-(58) are in agreement with Eqs. (47), (48) of Ref. [23].

As a consequence, the average transverse momentum squared, $\langle\, \boldsymbol{k}_T^2(x) \,\rangle_q$, can be easily obtained:

$$\langle\, \boldsymbol{k}_T^2(x) \,\rangle_q \; = \; \frac{M^2 x^2}{q(x)} \int_x^1 {\rm d}y \, \frac{q(y)}{y} \, . \tag{59}$$

For the case of polarized DIS the result is slightly more complicated. Given the functional form of $\Delta q(x_{\rm Bj})$ we can obtain $\varphi_8(\eta)$ via Eq. (40), in agreement with Eq. (42) of Ref. [20]:

$$\varphi_8(\eta) = -\frac{1}{\eta^3} \left\{ 3\, \Delta q(\eta) - \eta \, \frac{\rm d}{{\rm d}\eta} \, \Delta q(\eta) + 2 \int_\eta^1 \frac{{\rm d}\eta'}{\eta'} \, \Delta q(\eta') \right\} . \tag{60}$$

The transverse momentum dependent polarized parton density is then given, via Eq. (35) and the relation between η and $\boldsymbol{k}_T^2$ in Eq. (58), by

$$\Delta q(x, \boldsymbol{k}_T^2) = \frac{1}{\pi M^2} \left[\frac{\boldsymbol{k}_T^2}{xM^2} - x \right] \varphi_8\big(x + \frac{\boldsymbol{k}_T^2}{xM^2}\big) \, \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2] \, , \tag{61}$$

with

$$\int \mathrm{d}^2\boldsymbol{k}_T\, \Delta q(x, \boldsymbol{k}_T^2) = \Delta q(x)\,. \tag{62}$$

Just as for the unpolarized density, we see, via Eq. (60), that the $\boldsymbol{k}_T^2$ dependence of $\Delta q(x, \boldsymbol{k}_T^2)$ is completely determined by the x_{Bj} dependence of the collinear polarized quark density $\Delta q(x_{\mathrm{Bj}})$.

Of great interest at present is the transversely polarized quark density [24] (also referred to as the transversity, see e.g. Ref. [30] for a review) which is concerned with quarks transversely polarized along the y-direction inside a nucleon transversely polarized along OY:

$$\begin{aligned} \Delta_T q(x,\, \boldsymbol{k}_T) &\equiv q_{s_y/S_Y}(x,\, \boldsymbol{k}_T) - q_{-s_y/S_Y}(x,\, \boldsymbol{k}_T) \\ &= \frac{\eta}{\pi M^2}\, \varphi_{11}\big(x + \frac{\boldsymbol{k}_T^2}{xM^2}\big)\, \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2]\, \cos\phi. \end{aligned} \tag{63}$$

Here we specify the parton distributions using the physically motivated probabilistic notation of Ref. [29], referring to the polarization of a parton, whose spin direction (in its helicity frame whose axes are labelled ox, oy, oz [31]) is indicated by $s = s_x, s_y$ for transverse polarization, inside a nucleon moving along OZ in some fixed frame with axes labelled OX, OY, OZ (it could be the laboratory frame or the c.m. frame of the γ^*-nucleon collision), whose spin direction is indicated by $S = S_X, S_Y$ for transverse polarization, namely

$$\Delta q_{s/S}(x,\, \boldsymbol{k}_T) \equiv q_{s/S}(x,\, \boldsymbol{k}_T) - q_{-s/S}(x,\, \boldsymbol{k}_T)\,. \tag{64}$$

The connection with the more formal notation of Ref. [26] will be given in section IV A (see also appendix C of Ref. [29]). In Eq. (63) ϕ is the azimuthal angle of $\boldsymbol{k}_T$ in the fixed frame.

Note that

$$\int \mathrm{d}^2\boldsymbol{k}_T\, \Delta_T q(x, \boldsymbol{k}_T) = 0\,. \tag{65}$$

This is because $\Delta_T q(x, \boldsymbol{k}_T)$ corresponds to having the quark polarized along oy perpendicular to *its* momentum and not along OY, perpendicular to the nucleon's momentum. The $\boldsymbol{k}_T$-dependent density which does integrate to the usual collinear transversely polarized or transversity density is one in which the quark is polarized along the direction OY in the fixed frame:

$$\begin{aligned} \Delta q_{s_Y/S_Y}(x, \boldsymbol{k}_T^2) &= \Delta_T' q(x, \boldsymbol{k}_T^2) \qquad \text{in the notation of Ref. [30]} \\ &= h_1(x, \boldsymbol{k}_T^2) \qquad\quad \text{in the notation of Ref. [26]} \end{aligned} \tag{66}$$

so that

$$\Delta_T q(x) = \int \mathrm{d}^2\boldsymbol{k}_T\, h_1(x, \boldsymbol{k}_T^2)\,. \tag{67}$$

One finds

$$\Delta_T' q(x, \boldsymbol{k}_T^2) \equiv h_1(x,\, \boldsymbol{k}_T^2) = \frac{x}{\pi\, M^2}\, \varphi_{11}\big(x + \frac{\boldsymbol{k}_T^2}{xM^2}\big)\, \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2]\,. \tag{68}$$

It follows that

$$\Delta_T q(x, \boldsymbol{k}_T) = \frac{\eta}{x}\, h_1(x, \boldsymbol{k}_T^2)\, \cos\phi\,. \tag{69}$$

Moreover, from Eq. (67) one can show that

$$\varphi_{11}(\eta) = \frac{1}{\eta^2}\left\{\frac{2}{\eta}\Delta_T q(\eta) - \frac{\mathrm{d}}{\mathrm{d}\eta}\Delta_T q(\eta)\right\} = -\frac{\mathrm{d}}{\mathrm{d}\eta}\Big[\frac{\Delta_T q(\eta)}{\eta^2}\Big]\,. \tag{70}$$

Thus, via Eqs. (68), (69), the $\boldsymbol{k}_T$ dependence of both $h_1(x,\, \boldsymbol{k}_T^2)$ and $\Delta_T q(x, \boldsymbol{k}_T)$ is completely determined by the functional form of the standard collinear transversity density $\Delta_T q(x)$.

It is an intriguing question whether, in the real case of $g \neq 0$, it is possible to have a factorized form $F(x)G(\boldsymbol{k}_T^2)$ for the parton densities. Aside from the more complex tensorial structure when $g \neq 0$, the functional dependence will be controlled by expressions of the form

$$\int \mathrm{d}k^2\; \Phi\left(k^2, x + \frac{\boldsymbol{k}_T^2 + k^2}{xM^2}\right)\,,$$

where the range of k^2 is limited to a small region around $k^2 = 0$. While it seems unlikely that this could lead to a factorized form, we have been unable to demonstrate this mathematically.

A. Other transverse momentum dependent quark densities

In this subsection we shall give expressions for several other TMD quark densities, which, while playing no role in the $\boldsymbol{k}_T$-integrated hadronic correlator for DIS, are important in other processes, like semi-inclusive DIS or Drell-Yan dilepton production, and have been widely used in the literature [11, 30].

Related to A_8 we consider quarks polarized longitudinally (i.e. with helicity $\pm 1/2$) inside a transversely polarized nucleon (here it is irrelevant which transverse direction is involved). Then (see also Ref. [26])

$$\Delta q_{s_z/\boldsymbol{S}_T}(x,\, \boldsymbol{k}_T) \equiv q_{+/\boldsymbol{S}_T}(x,\, \boldsymbol{k}_T) - q_{-/\boldsymbol{S}_T}(x,\, \boldsymbol{k}_T) = \frac{\boldsymbol{k}_T \cdot \boldsymbol{S}_T}{M}\, g_{1T}\,, \tag{71}$$

with

$$g_{1T}(x,\, \boldsymbol{k}_T^2) = -\,\frac{2}{\pi M^2}\ \varphi_8\big(x + \frac{\boldsymbol{k}_T^2}{xM^2}\big)\, \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2]\,. \tag{72}$$

For the quark polarized in the x-direction, we have

$$\begin{aligned}
\Delta q_{s_x/S_Y}(x,\, \boldsymbol{k}_T) \ &\equiv\ q_{s_x/S_Y}(x,\, \boldsymbol{k}_T) - q_{-s_x/S_Y}(x,\, \boldsymbol{k}_T) \\
&=\ \Big\{ h_1(x,\, \boldsymbol{k}_T^2) + \frac{\boldsymbol{k}_T^2}{2M^2} h_{1T}^{\perp}(x,\, \boldsymbol{k}_T^2) \Big\} \sin\phi\,, \\
&=\ -\,\frac{1}{\pi M^2}\Big[\frac{\boldsymbol{k}_T^2}{xM^2} - x\Big]\, \varphi_{11}\big(x + \frac{\boldsymbol{k}_T^2}{xM^2}\big)\, \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2]\, \sin\phi\,.
\end{aligned} \tag{73}$$

One also finds that

$$h_{1T}^{\perp}(x, \boldsymbol{k}_T^2) = -\,\frac{2}{\pi x\, M^2}\ \varphi_{11}\big(x + \frac{\boldsymbol{k}_T^2}{xM^2}\big)\, \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2]\,. \tag{74}$$

For the function $h_{1T}(x, \boldsymbol{k}_T^2)$ defined via

$$h_1(x, \boldsymbol{k}_T^2) \equiv h_{1T}(x, \boldsymbol{k}_T^2) + \frac{\boldsymbol{k}_T^2}{2\, M^2}\, h_{1T}^{\perp}(x, \boldsymbol{k}_T)\,, \tag{75}$$

we find

$$h_{1T}(x, \boldsymbol{k}_T^2) = \frac{1}{\pi\, M^2}\Big[x + \frac{\boldsymbol{k}_T^2}{xM^2}\Big]\, \varphi_{11}\big(x + \frac{\boldsymbol{k}_T^2}{xM^2}\big)\, \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2]\,. \tag{76}$$

Finally, for a quark polarized along the x-direction inside a longitudinally polarized nucleon,

$$\Delta q_{s_x/S_Z}(x,\, \boldsymbol{k}_T^2) \equiv q_{s_x/+}(x,\, \boldsymbol{k}_T^2) - q_{-s_x/+}(x,\, \boldsymbol{k}_T^2) = \frac{|\boldsymbol{k}_T|}{M}\, h_{1L}^{\perp}(x, \boldsymbol{k}_T^2)\,, \tag{77}$$

with

$$h_{1L}^{\perp}(x, \boldsymbol{k}_T^2) = -\,\frac{2}{\pi\, M^2}\ \varphi_{11}\big(x + \frac{\boldsymbol{k}_T^2}{xM^2}\big)\, \theta[x(1-x)\, M^2 - \boldsymbol{k}_T^2]\,. \tag{78}$$

Note that for consistency in the parton model, with $g = 0$, we have, in agreement with the results of EZTS,

$$\frac{x}{\eta}\, h_{1T}(x, \boldsymbol{k}_T^2) = -\frac{x}{2}\, h_{1L}^{\perp}(x, \boldsymbol{k}_T^2) = -\frac{x^2}{2}\, h_{1T}^{\perp}(x, \boldsymbol{k}_T^2) = h_1(x, \boldsymbol{k}_T^2)\,. \tag{79}$$

V. BOUNDS ON THE COLLINEAR DENSITIES

In attempting to extract information on the collinear polarized density $\Delta q(x)$ and the transversity density $\Delta_T q(x)$ by fitting data using parameterized forms for the densities, it is usual to impose the positivity bound

$$|\Delta q(x)| \le q(x) \tag{80}$$

on $\Delta q(x)$, and the Soffer bound [12]

$$|\Delta_T q(x)| \le \frac{1}{2}\, [q(x) + \Delta q(x)] \tag{81}$$

on $\Delta_T q(x)$. However, it turns out that while these bounds are *necessary* they are not *sufficient*.

A. The new positivity bound

The reason why Eq. (80) is not sufficient is that from a probabilistic point of view one should expect

$$|\Delta q(x, \boldsymbol{k}_T^2)| \leq q(x, \boldsymbol{k}_T^2) \;, \tag{82}$$

and while Eq. (82) implies Eq. (80), the reverse is not true. Actually, as we shall now show, it turns out that it is sufficient to require just

$$|\Delta q(x, 0)| \leq q(x, 0) \;. \tag{83}$$

For then from Eq. (61)

$$\Delta q(x, 0) = -\frac{x}{\pi M^2}\, \varphi_8(x)\theta[x(1-x)\, M^2] \;, \tag{84}$$

and from Eq. (55)

$$q(x, 0) = \frac{1}{\pi M^2}\, \varphi_3(x)\theta[x(1-x)\, M^2] \;. \tag{85}$$

Since Eq. (83) must hold for all x we find that (of course $\varphi_3(\eta) \geq 0$)

$$|\varphi_8(\eta)| \leq \frac{\varphi_3(\eta)}{\eta} \;, \tag{86}$$

and this is sufficient to guarantee that Eq. (82) holds. Thus, we suggest that in parameterizing the collinear polarized density $\Delta q(x)$ one should perhaps impose the stronger bound

$$\left| 3\,\Delta q(x) - x\,\frac{\mathrm{d}}{\mathrm{d}x}\,\Delta q(x) + 2\int_x^1 \frac{\mathrm{d}x'}{x'}\,\Delta q(x') \right| \leq q(x) - x\,\frac{\mathrm{d}}{\mathrm{d}x}\,q(x) \;, \tag{87}$$

which follows from Eqs. (86), (60) and (56) and is in agreement with Eq. (48) of Ref. [20]; of course, one must have $q(x) - x\,q'(x) \geq 0$.

Notice that there is a further bound, originally derived in Ref. [20], Eq. (60),

$$\left| \Delta q(x) + 2\int_x^1 \frac{\mathrm{d}x'}{x'}\,\Delta q(x') \right| \leq q(x) \;, \tag{88}$$

which can be easily obtained from our Eqs. (39), (34), (53) and (86).

B. Reinterpretation of the Soffer bound

Soffer derived his bound on the collinear transversity density [12]

$$|\Delta_T q(x)| \leq \frac{1}{2}\,[q(x) + \Delta q(x)] \;, \tag{89}$$

by noticing the analogy between the quark correlator diagram and the diagram for the u-channel absorptive part of *forward* elastic quark-nucleon scattering. But collinear parton densities include all partons with transverse momentum $\boldsymbol{k}_T^2 \leq Q^2$ and thus do not consist of strictly forward moving partons. Hence, in the light of the existence of intrinsic $\boldsymbol{k}_T$, Soffer's analogy could be said to have been misinterpreted, and the correct bound should read

$$|\Delta_T q(x, 0)| \;\equiv\; |\Delta'_T q(x, 0)| \;\leq\; \frac{1}{2}\,[q(x, 0) + \Delta q(x, 0)] \;. \tag{90}$$

Now, from Eqs. (69), (68) we have

$$\Delta_T q(x, 0) \;\equiv\; \Delta'_T q(x, 0) \;=\; \frac{x}{\pi M^2}\,\varphi_{11}(x)\,\theta[x(1-x)\, M^2] \;, \tag{91}$$

and substituting this and Eqs. (84), (85) into Eq. (90) we obtain

$$|\varphi_{11}(\eta)| \leq \frac{1}{2\eta}\left[\varphi_3(\eta) - \eta\,\varphi_8(\eta)\right]. \tag{92}$$

We suggest that in parameterizing models for the collinear transversity density $\Delta_T q(x)$, the scalar functions $\varphi_{3,8,11}$ should be calculated in terms of $q(x)$, $\Delta q(x)$ and $\Delta_T q(x)$, via Eqs. (56), (60), (70), and, perhaps, should be forced to satisfy the bound in Eq. (92).

Finally, we have also explicitly verified that the bounds in Eqs. (86), (92) are equivalent to the bounds on the $\boldsymbol{k}_T$-dependent functions derived in Ref. [32], in the $g = 0$ limit.

C. Collinear and TMD quark distributions

As a simple and illustrative example of our results, we consider here some of the consequences of imposing Lorentz invariance on the $\boldsymbol{k}_T$-dependence of TMD parton distributions. We will start from a well-known set of unpolarized and longitudinally polarized distributions extracted within the standard collinear approach, namely the GRV98 [33] and GRSV2000 [34] sets respectively, and show the crucial role played by the new bounds. Since we are working in the $g = 0$ approximation, we will consistently consider only u and d quark distributions, neglecting sea quarks. Moreover, since we are here mainly interested in the intrinsic $\boldsymbol{k}_T$ dependence, we choose as reference scale a relatively low one, $Q^2 = 2$ GeV^2, in order to exclude large effects due to perturbative evolution. From these parameterizations for $q(x, Q^2)$ and $\Delta q(x, Q^2)$ we therefore generate, via Eq. (57) and Eqs. (60), (61) the $\boldsymbol{k}_T$-dependent distributions $q(x, \boldsymbol{k}_T^2, Q^2)$ and $\Delta q(x, \boldsymbol{k}_T^2, Q^2)$.

Concerning transversity, we have to recall that at present no parameterization in the standard collinear approach is available. Double transverse spin asymmetries, A_{TT}, in the Drell-Yan processes $p^\uparrow p^\uparrow \to \ell^+\ell^- + X$, or $p^\uparrow \bar{p}^\uparrow \to \ell^+\ell^- + X$, are considered the best tool to this end, but are unfortunately still out of reach for present experimental setups. In the meantime, the study of transverse single spin asymmetries in polarized SIDIS processes in the TMD approach has been found to be the most promising way to get information on the unintegrated, $\boldsymbol{k}_T$-dependent transversity distribution [35]. We will therefore utilize in our discussion the first ever parameterization available for the transversity of u and d quarks obtained by fitting, in the TMD approach, the so-called Collins azimuthal asymmetries measured in SIDIS pion and kaon production by the HERMES and COMPASS experiments. We have to stress, however, that this parameterization has been obtained by assuming a factorized x and $\boldsymbol{k}_T$-dependence for the TMD distributions, which is at variance with our findings in the simplified, $g = 0$, limit. Given the still poor knowledge of the transversity distribution, we believe however that, despite this inconsistency, it is of interest to investigate the consequences of

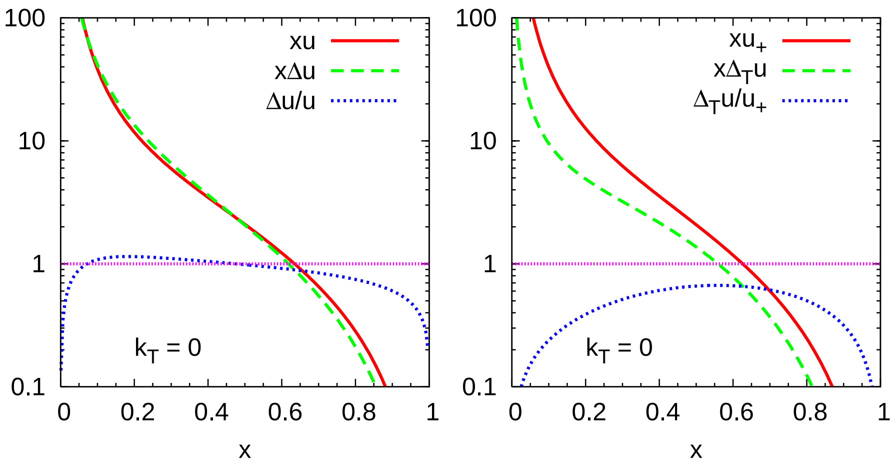

FIG. 3: Left panel: unpolarized (solid line), longitudinally polarized (dashed line) distributions and their ratio (dotted line) for up quarks as a function of x at $|\boldsymbol{k}_T| = 0$ and $Q^2 = 2$ GeV^2, obtained via Eq. (57) and Eqs. (60), (61) starting from $u(x)$ [GRV98] and $\Delta u(x)$ [GRSV2000]. Note the violation of the positivity bound, Eq. (83). Right panel: Soffer-type bound, $u_+ \equiv (u + \Delta u)/2$, (solid line), transversity distribution (dashed line) and their ratio (dotted line) for up quarks as a function of x at $|\boldsymbol{k}_T| = 0$ and $Q^2 = 2$ GeV^2, obtained via Eqs. (70), (91) starting from $\Delta_T u(x)$ of Ref. [36].

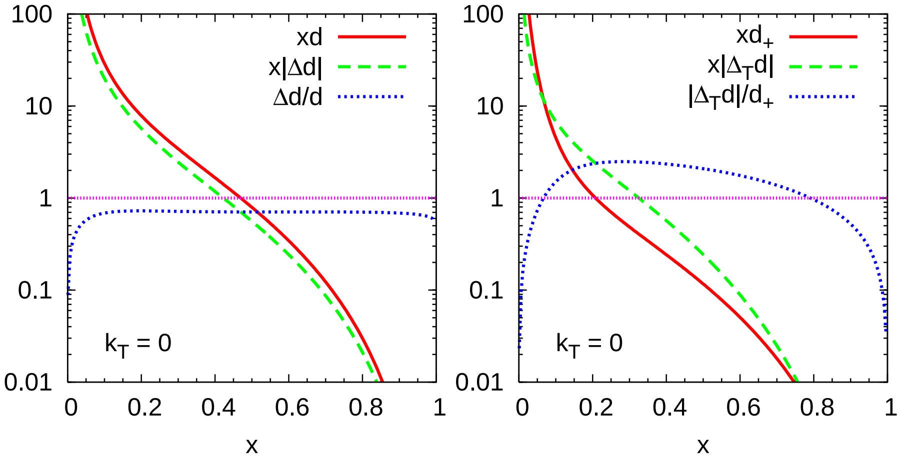

FIG. 4: Same as in Fig. 3, but for down quarks. Note, in the right panel, the strong violation of the Soffer-type bound, Eq. (90).

our approach. Starting from the most updated parameterization of the transversity distribution for valence quarks, Ref. [36], and using Eqs. (70), (68), we therefore generate the $g = 0$, $\boldsymbol{k}_T$-dependent distribution $\Delta'_T q(x, \boldsymbol{k}_T^2, Q^2)$.

Our results are summarized in Figs. 3-7. In Fig. 3, left panel, we show the unpolarized and longitudinally polarized u quark distributions, and their ratio, as a function of x at fixed $|\boldsymbol{k}_T| = 0$. Although the collinear parton distributions adopted here fulfill the usual positivity bound, Eq. (80), we see that the stronger positivity bound of Eq. (83) is indeed violated for some x values. This is most clearly seen by looking at the ratio $\Delta u/u$ (dotted line), where the violation reaches about 15%. In Fig. 3, right panel, we also show, again for u quarks, the transversity distribution $\Delta_T u$ and the positive-helicity distribution $u_+ = (u + \Delta u)/2$ and their ratio (related to the Soffer bound), as a function of x at fixed $|\boldsymbol{k}_T| = 0$. In this case also the stronger, Soffer-type, bound, Eq. (90), is fulfilled.

Similar results are shown in Fig. 4 for d quarks. This time, while the stronger positivity bound derived in this section is always fulfilled, the Soffer-type bound, Eq. (90), is instead violated over a large range of x values.

In Fig. 5 we show, again for $\Delta u(x, \boldsymbol{k}_T^2)$ in the left panel and for $\Delta'_T u(x, \boldsymbol{k}_T^2)$ in the right panel, the $|\boldsymbol{k}_T|$ dependence

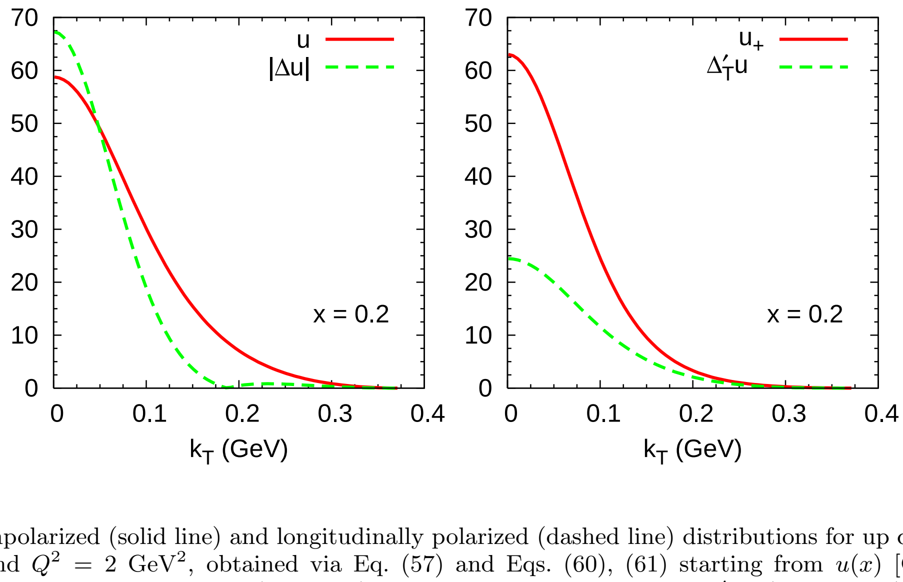

FIG. 5: Left panel: unpolarized (solid line) and longitudinally polarized (dashed line) distributions for up quarks as a function of $|\boldsymbol{k}_T|$ at $x = 0.2$ and $Q^2 = 2$ GeV^2, obtained via Eq. (57) and Eqs. (60), (61) starting from $u(x)$ [GRV98] and $\Delta u(x)$ [GRSV2000]. Right panel: Soffer-type bound (solid line) and transversity distribution, $\Delta'_T u$, (dashed line) for up quarks, as a function of $|\boldsymbol{k}_T|$ at $x = 0.2$ and $Q^2 = 2$ GeV^2, obtained via Eqs. (68), (70) starting from $\Delta_T u(x)$ of Ref. [36].

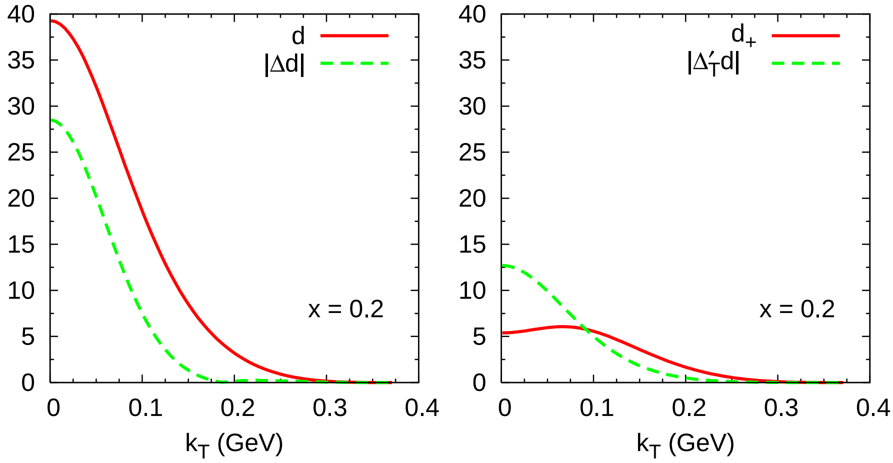

FIG. 6: Same as in Fig. 5, but for down quarks.

of the u-quark distributions at a fixed value of x in the valence region, $x = 0.2$. For the longitudinal distribution, we see that the violation of the stronger positivity bound at $|\boldsymbol{k}_T| = 0$ persists up to some relatively large values of $|\boldsymbol{k}_T|$. A similar behaviour should result for any value of x in the region of violation of the positivity bound at $|\boldsymbol{k}_T| = 0$, shown in the left panel of Fig. 3. Notice also the node in Δu, which is a general feature of Δq coming from Eq. (61). Finally, notice from the right panel that, as expected, the validity of the Soffer-type bound at $|\boldsymbol{k}_T| = 0$ guarantees its validity at any allowed value of $|\boldsymbol{k}_T|$. Analogous results for d quarks are presented in Fig. 6, and similar comments hold on the violation, this time, of the Soffer-type bound.

For completeness, in Fig. 7 we show the resulting mean transverse momentum squared, $\langle\, \boldsymbol{k}_T^2(x)\,\rangle_q$, Eq. (59), as a function of x, for u and d quarks, at the same reference scale, $Q^2 = 2$ GeV^2. We briefly note that the shape of $\langle\, \boldsymbol{k}_T^2(x)\,\rangle_q$ is quite reasonable and approximately flavour-independent, although its size is considerably smaller than expected from available phenomenological analyses within the TMD approach [37, 38].

This is perhaps a hint that the larger values of $\langle\, \boldsymbol{k}_T^2\,\rangle$ needed in the phenomenological analyses may be due to hard

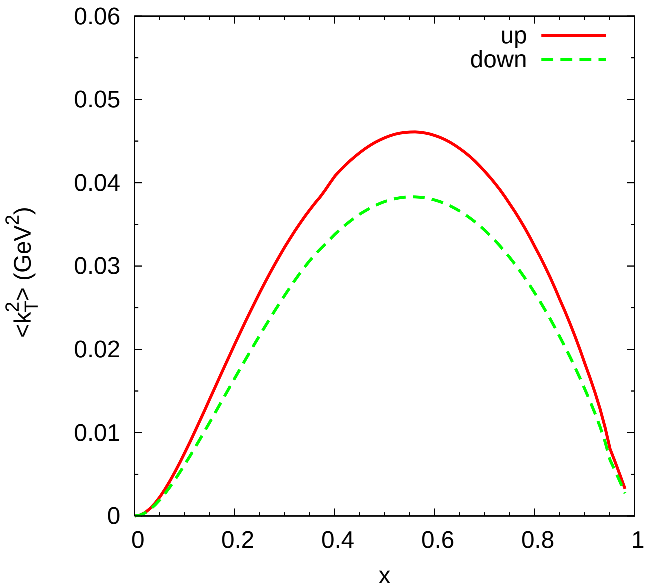

FIG. 7: Mean transverse momentum squared, $\langle\, \boldsymbol{k}_T^2(x)\,\rangle_q$, Eq. (59), as a function of x, for up (solid line) and down (dashed line) quarks at $Q^2 = 2$ GeV^2.

gluon emission and may therefore not be a measure of the size of the genuinely intrinsic transverse momentum. It would be interesting to construct models based upon such a picture.

VI. CONCLUSIONS

We have shown that the simple parton model has a very rich structure when due care is given to its Lorentz properties. On the basis that *kinematic* relations should not depend on the value of the strong coupling g, and thus should be valid when $g = 0$, we have been able to give a simple derivation of the exact target mass corrections for polarized DIS, without recourse to the operator product expansion, and have verified that the Wandzura-Wilczek relation holds when target mass corrections are included. We have also demonstrated that that there is a surprising and intimate connection between the intrinsic $\boldsymbol{k}_T$ dependence of parton densities and the functional form of the $\boldsymbol{k}_T$-integrated collinear densities. Indeed the $\boldsymbol{k}_T$ dependence can actually be *derived* from the collinear densities. Using some published versions of the collinear densities we have studied the implications of this relationship and found that our $\boldsymbol{k}_T$-dependent distributions, thus derived, are much narrower than those used in phenomenological work i.e. our $\langle\, \boldsymbol{k}_T^2 \,\rangle$ is much smaller. This suggests that some of the $\boldsymbol{k}_T$ dependence usually described as intrinsic, may, in fact, be due to gluon radiation.

One outcome of our work is the clear demonstration that the factorized form $g(x)f(\boldsymbol{k}_T^2)$ often used in phenomenological studies, is untenable, *at least when* $g = 0$. While not proved, it seems likely that this result is quite general, though, as commented on, a factorized form does seem a reasonable starting point for the analysis of the presently available data. In addition, because of this interrelationship between $\boldsymbol{k}_T$ and x dependence, we have been able to derive new positivity and Soffer-type bounds on the collinear densities, which are stronger than the those usually utilized in DIS and SIDIS analyses. We have shown numerically that some of the published collinear parton densities, which satisfy the traditional bounds, do not always satisfy these stronger bounds. The particular case of the down-quark transversity may be of some interest.

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