On the controversy concerning the definition of quark and gluon angular momentum

Elliot Leader

Imperial College London

Important question: how are the momentum and angular momentum of a nucleon built up from the momenta and angular momenta of its constituents?

Background

• Controversy in QCD : how to split the total angular momentum into separate quark and gluon components
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- Ji stresses: gauge invariant operators; covariance; local operators
- Chen at al and Wakamatsu: don't like Ji; don't like any previous theory; claim even in QED the traditional, decades-old identification of electron and photon angular momentum is incorrect
- Different results for momentum and angular momentum carried by quarks and gluons e.g. as $\mu^2 \to \infty$

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Since problem already arises in QED, will illustrate via QED

There are four versions of $\boldsymbol{J}$
Canonical (can), Bellinfante (bel)= Ji, Chen at al (chen), Wakamatsu (wak)

$$
\begin{aligned}
\boldsymbol{J}_{can} &= \int d^3x\, \psi^\dagger \boldsymbol{\gamma}\gamma_5 \psi + \int d^3x\, \psi^\dagger [\boldsymbol{x} \times (-i\boldsymbol{\nabla})]\psi \\
&+ \int d^3x\, (\boldsymbol{E} \times \boldsymbol{A}) + \int d^3x\, E^i [\boldsymbol{x} \times \boldsymbol{\nabla} A^i] \\
&= \boldsymbol{S}_{can}(el) + \boldsymbol{L}_{can}(el) + \boldsymbol{S}_{can}(\gamma) + \boldsymbol{L}_{can}(\gamma)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{J}_{bel} &= \int d^3x\, \psi^\dagger \boldsymbol{\gamma}\gamma_5 \psi + \int d^3x\, \psi^\dagger [\boldsymbol{x} \times (-i\boldsymbol{D})]\psi \\
&+ \int d^3x\, \boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B}) \\
&= \boldsymbol{S}_{bel}(el) + \boldsymbol{L}_{bel}(el) + \boldsymbol{J}_{bel}(\gamma)
\end{aligned}
$$

Note: $\boldsymbol{J}_{bel}(\gamma)$ NOT split into spin and orbital parts.

$$
\begin{aligned}
\boldsymbol{J}_{chen} &= \int d^3x\, \psi^\dagger \boldsymbol{\gamma}\gamma_5 \psi + \int d^3x\, \psi^\dagger [\boldsymbol{x} \times (-i\boldsymbol{D}_{pure})]\psi \\
&+ \int d^3x\, (\boldsymbol{E} \times \boldsymbol{A}_{phys}) + \int d^3x\, E^i [\boldsymbol{x} \times \boldsymbol{\nabla} A^i_{phys}] \\
&= \boldsymbol{S}_{ch}(el) + \boldsymbol{L}_{ch}(el) + \boldsymbol{S}_{ch}(\gamma) + \boldsymbol{L}_{ch}(\gamma)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{J}_{wak} &= \int d^3x\, \psi^\dagger \boldsymbol{\gamma}\gamma_5 \psi + \int d^3x\, \psi^\dagger [\boldsymbol{x} \times (-i\boldsymbol{D})]\psi \\
&+ \int d^3x\, (\boldsymbol{E} \times \boldsymbol{A}_{phys}) \\
&+ \left[\int d^3x\, E^i (\boldsymbol{x} \times \boldsymbol{\nabla} A^i_{phys}) + \int d^3x\, \psi^\dagger (\boldsymbol{x} \times e\boldsymbol{A}_{phys})\psi\right] \\
&= \boldsymbol{S}_{wak}(el) + \boldsymbol{L}_{wak}(el) + \boldsymbol{S}_{wak}(\gamma) + \boldsymbol{L}_{wak}(\gamma)
\end{aligned}
$$

In this version the very last term $\int d^3x\, \psi^\dagger (\boldsymbol{x} \times e\boldsymbol{A}_{phys})\psi$ has been shifted from Chen et al's electron orbital term to the photon's orbital angular momentum.

As usual $D^\mu = \partial^\mu - ieA^\mu$

Chen et al: $\boldsymbol{A} = \boldsymbol{A}_{phys} + \boldsymbol{A}_{pure}$

$$\boldsymbol{\nabla}.\boldsymbol{A}_{phys} = 0 \qquad \boldsymbol{\nabla} \times \boldsymbol{A}_{pure} = 0$$

Corresponds exactly to what is usually called the transverse $\boldsymbol{A}_\perp$ and longitudinal $\boldsymbol{A}_\parallel$ parts respectively

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• $\boldsymbol{A}_{phys}$ is not a local field:

$$\boldsymbol{A}_{phys} = \boldsymbol{A} - \frac{1}{\nabla^2}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})$$

Which is “correct”?

What is the criterion for deciding?

Similar differences in definitions of linear momentum. Asymptotically what fraction of total momentum is carried by gluons?

Ji: $\frac{16}{16+3n_f} \simeq 1/2 \qquad$ for $n_f = 5$

Chen et al: $\frac{8}{8+6n_f} \simeq 1/5 \qquad$ for $n_f = 5$!

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Beware quantization conditions! Bellinfante, as used, does not correspond to covariant quantization.

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Impossible. Cannot be checked!

Will only have time to discuss some aspects of these problems

Many of the problems involved also apply to **linear momentum**.

Also many apply in **QED**

Much simpler, therefore illustrate them using linear momentum in QED.

The momentum operator in gauge-invariant theories

Theory invariant under translations; Noether construction, from classical Lagrangian; canonical e-m density $t^{\mu\nu}_{can}(x)$. A conserved density, generally not symmetric under $\mu \leftrightarrow \nu$.

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Canonical total linear momentum operator P^j_{can}

$$P^j_{can} = \int d^3x \; t^{0j}_{can}(x)$$

independent of time.

Canonical momentum operator as generator of translations

Classically : P^j_{can} generates spatial translations.
Quantum theory: check correct commutation relations with fields i.e. for any field $\phi(x)$

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But Equal Time Commutators (ETC) fixed in quantizing theory. Thus can check because P^j_{can} independent of time. Take time variable of fields in P^j_{can} to coincide with time variable in $\phi(x) \equiv \phi(t, \boldsymbol{x})$.

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Will be crucial when discussing division of total momentum into contributions from different fields .

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Differs from $t^{\mu\nu}_{can}(x)$ by a divergence term:

$$t^{\mu\,\nu}_{bel}(x) = t^{\mu\,\nu}_{can}(x) + \frac{1}{2}\partial_\rho[H^{\rho\mu\nu} - H^{\mu\rho\nu} - H^{\nu\rho\nu}]$$

where $H^{\rho\mu\nu} = -H^{\rho\nu\mu}$ and is a local operator.

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For a classical *c-number* field it is meaningful to argue that the field vanishes at infinity. Much less obvious what this means for a quantum operator.

Is it safe to throw away integral of divergence ??

It had better be, otherwise a catastrophe

Would find that P^j does not commute with itself !

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$$A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x)$$

where $\Lambda(x)$ is a c-number field satisfying $\Box\Lambda(x) = 0$ and vanishing at infinity.

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Let F be the generator of gauge transformations, so that

$$i[F, A^\mu(x)] = \partial^\mu \Lambda(x)$$

Consider the Jacobi identity

$$[F,\, [P^\mu, A^\nu]] + [A^\nu,\, [F, P^\mu]] + [P^\mu,\, [A^\nu, F]] = 0$$

Now $[P^\mu,\, [A^\nu, F]] = 0$ since $[A^\nu, F]$ is a c-number. Thus

$$[[F, P^\mu],\, A^\nu] = [F,\, [P^\mu, A^\nu]] \qquad (\alpha)$$

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$$[[F,P^\mu],\,A^\nu]=[F,\,[P^\mu,A^\nu]] \qquad (\alpha)$$

Since P^μ are generators of translations, $i[P^\mu,A^\nu]=\partial^\mu A^\nu$

Thus the RHS of Eq. (α) becomes

$$[F,\,[P^\mu,A^\nu]]=-i\partial^\mu[F,A^\nu(x)]=-\partial^\mu\partial^\nu\Lambda(x)\neq 0$$

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Hence from Eq. (α)

$$[[F,P^\mu],\,A^\nu]\neq 0$$

so that P^μ is not gauge invariant.

However, lack of gauge invariance of no physical significance.

Example, covariantly quantized QED: show that the matrix element of P^j_{can} between any normalizable physical states, unaffected by gauge changes in the operator.

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Lautrup-Nakanishi Lagrangian density: combination of the Classical Lagrangian ($Clas$) and a Gauge Fixing part (Gf) $\qquad \mathcal{L} = \mathcal{L}_{Clas} + \mathcal{L}_{Gf}$

$$\mathcal{L}_{Clas} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} [\bar{\psi}(i \not{\partial} - m + e \not{A})\psi + \text{h.c.}]$$

$$\mathcal{L}_{Gf} = B(x)\, \partial_\mu A^\mu(x) + \frac{\mathrm{a}}{2} B^2(x)$$

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Theory invariant under c-number infinitesmal gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda(x) \qquad \psi \to \psi + ie\Lambda\psi$$

while $B(x)$ is unaffected by gauge transformations.

$$\text{Generator}\, F = \int d^3x \, [(\partial_0 B)\Lambda - B\partial_0\Lambda + \partial_j(F^{0j}\Lambda)].$$

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Generator $F = \int d^3x\, [(\partial_0 B)\Lambda - B\partial_0\Lambda + \partial_j(F^{0j}\Lambda)] \qquad (\beta)$

Physical states $|\Psi\rangle$ of the theory defined to satisfy

$$B^{(+)}(x)|\Psi\rangle = 0$$

$$B(x) = B^{(+)}(x) + B^{(-)}(x) \qquad B^{(-)}(x) = [B^{(+)}]^\dagger(x)$$

Thus for arbitrary physical states

$$\langle \Psi' | B(x) | \Psi \rangle = 0 \qquad (\gamma)$$

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Theorem Physical matrix elements of P^j are invariant under gauge transformations.

Proof Consider the general physical matrix element

$$\langle \Psi' | P^j | \Psi \rangle = \int d^3\boldsymbol{p}\, d^3\boldsymbol{p}'\, \phi(\boldsymbol{p})\, \phi'(\boldsymbol{p}')\, \langle \boldsymbol{p}' | P^j | \boldsymbol{p} \rangle$$

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Change induced in $\langle \boldsymbol{p}' | P^j | \boldsymbol{p} \rangle$ is $\langle \boldsymbol{p}' | i[F, P^j] | \boldsymbol{p} \rangle$.

First two terms in F, Eq. (β), give zero because of Eq. (γ) and the fact that Λ is a c-number.

Change induced by the divergence term is

$$\begin{aligned}\int d^3x \langle \boldsymbol{p}'| i[\partial_k(F^{0k}\Lambda), P^j]|\boldsymbol{p}\rangle &= (p'-p)^j[(p^0-p'^0)\,\langle \boldsymbol{p}'|A^k(0)|\boldsymbol{p}\rangle \\ &- (p-p')^k \langle \boldsymbol{p}'|A^0(0)|\boldsymbol{p}\rangle] \\ &\times \int d^3x \partial_k[\Lambda(x)\, e^{i(p-p')\cdot x}]\end{aligned}$$

which vanishes after the spatial integration because $\Lambda(x)$ vanishes at infinity.

Hence $\langle \Psi'|P^j|\Psi\rangle$ is indeed invariant under gauge transformations.

The matrix elements of the angular momentum operators

A subtle problem: many incorrect statements in the literature

We consider a nucleon with 4-momentum p^{μ} and covariant spin vector S corresponding to some specification of its spin state e.g. helicity, transversity or spin along the Z-axis i.e. a nucleon in state $|p, S\rangle$.

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We require an expression for the expectation value of the angular momentum in this state i.e. for $\langle p, S|\boldsymbol{J}|p, S\rangle$

i.e. we require an expression in terms of p and S. This can then be used to relate the expectation value of $\boldsymbol{J}$ for the nucleon to the angular momentum carried by its constituents.

The traditional approach:

In every field theory there is an expression for the angular momentum density operator. The angular momentum operator $\boldsymbol{J}$ is then an integral over all space of this density.

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Typically the angular momentum density involves the energy-momentum tensor density $t^{\mu\nu}(x)$ in the form e.g.

$$\boldsymbol{J}_z = \boldsymbol{J}^3 = \int dV \, [x t^{02}(\boldsymbol{x}) - y t^{01}(\boldsymbol{x})]$$

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$$= \int dV x \langle p, S| e^{i\boldsymbol{P}.\boldsymbol{x}} t^{02}(0) e^{-i\boldsymbol{P}.\boldsymbol{x}} |p, S\rangle$$

Now the nucleon is in an eigenstate of momentum, so $\boldsymbol{P}$ acting on it just becomes $\boldsymbol{p}$. The numbers $e^{i\boldsymbol{p}.\boldsymbol{x}}e^{-i\boldsymbol{p}.\boldsymbol{x}}$ cancel out and we are left with:

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The problem is an old one: In ordinary QM plane wave states give infinities

The solution is an old one: Build a wave packet, a superposition of physical plane wave states

This involves studying non-forward matrix elements and then taking the forward limit .

So we need expressions for matrix elements like

$$\langle\, p + \Delta/2; S \,|\, t^{\mu\nu}(0) \,|\, p - \Delta/2; S \,\rangle$$

Points which are handled incorrectly in the literature:

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Points which are handled incorrectly in the literature:

1) For a PHYSICAL wave packet the physical requirement on the covariant polarization vector i.e.
$S \cdot (p \pm \Delta/2) = 0$ implies $S \cdot \Delta = 0$.

2)

$$\langle\, p + \Delta/2; S \,|\, t^{\mu\nu}(0) \,|\, p - \Delta/2; S \,\rangle$$

does NOT transform as a tensor!!

To see this think of electromagnetic form factors:

$$\langle p', S | j^{\mu}_{em} | p, S \rangle$$

We cannot say: this transforms like a 4-vector, and therefore we can express it terms of vectors built from p, p', S

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We have to first factor out the Dirac spinors

$$\bar{u}(p')[\gamma^{\mu}\, F_1 + \frac{i\sigma^{\mu\nu} q_{\nu}}{2m}\, F_2] u(p)$$

Relating the matrix elements of $\boldsymbol{J}$ to the matrix elements of $t^{\mu\nu}$, using wave packets is tortuous. We shall come back to that later.

The problem of defining separate quark and gluon momenta

Two separate issues:(1) general problem of how to define the separate momenta for a system of interacting particles, (2) more specific to gauge theories and includes the issue of splitting the angular momentum of a gauge particle into a spin and orbital part.

(1) The general problem: System of interacting particles E and F. Split the total momentum into two pieces

$$P^j = P_E^j + P_F^j$$

which we associate with the momentum carried by the individual particles E and F respectively.

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Note that this expression is totally misleading, and should be written

$$P^j = P^j_E(t) + P^j_F(t)$$

to reflect the fact that the particles exchange momentum as a result of their interaction.

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But there is no way we can check this, since $P^j_E(t)$ depends on t and, without solving the entire theory, we are only able to compute equal time commutators .

We suggest, therefore, that *the minimal requirement for identifying an operator* P_E^j *with the momentum carried by* E , is to demand that *at equal times*

$$i[P_E^j(t)\,,\phi^E(t,\boldsymbol{x})] = \partial^j\phi^E(t,\boldsymbol{x}).$$

Analogously, *for an angular momentum operator* M_E^{ij} *(* $J^i = \epsilon^{ijk} M^{jk}$*) we suggest* that at equal times

$$i[M_E^{ij}(t)\,,\phi_r^E(t,\boldsymbol{x})] = (x^i\partial^j - x^j\partial^i)\phi_r^E(t,\boldsymbol{x}) + (\Sigma^{ij})_r{}^s\phi_s^E(t,\boldsymbol{x})$$

where r and s are spinor or Lorentz labels and $(\Sigma^{ij})_r{}^s$ is the relevant spin operator.

Implications

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But, if we split P_{can} into $P_{can,\,E} + P_{can,\,F}$ and P_{bel} into $P_{bel,\,E} + P_{bel,\,F}$, then the integrands of $P_{can,\,E}$ and $P_{bel,\,E}$ do *not* differ by a spatial divergence.

Implications

For the **total** momentum there is no essential difference between P_{can} and P_{bel} since their integrands differ by the spatial divergence of a local operator.

But, if we split P_{can} into $P_{can,\,E} + P_{can,\,F}$ and P_{bel} into $P_{bel,\,E} + P_{bel,\,F}$, then the integrands of $P_{can,\,E}$ and $P_{bel,\,E}$ do *not* differ by a spatial divergence.

Hence $P_{can,\,E}$ and $P_{bel,\,E}$ do **not** generate the same transformation on $\phi^E(x)$, and similarly for F.

Since, by construction, $P_{can,\,E}$ and $P_{can,\,F}$ do generate the correct transformations on $\phi_E(x)$ and $\phi_F(x)$ respectively, we conclude that with the above minimal requirement we are forced to associate the momentum and angular momentum of E and F with the canonical version of the relevant operators.

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This disagrees with Ji, Chen et al and Wakamatsu, but agrees with Jaffe and Manohar.

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In fact, no contradiction in the special case of the *longitudinal* components of the momentum and angular momentum.

¿From gauge invariant expression for the unpolarized quark number density $q(x)$ (including Wilson line operator) one finds

$$\int_0^1 dx\, x\,[\,q(x)+\bar{q}(x)\,] = \frac{i}{4(P^+)^2}\langle\, P\,|\,\bar{\psi}(0)\,\gamma^+\,\overleftrightarrow{D}^+\,\psi(0)\,|\,P\,\rangle$$

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But the quark part of $t^{\mu\nu}_{bel}(qG)$ is given by

$$t^{\mu\nu}_{q,\,bel}(z) = \frac{i}{4}[\bar{\psi}(z)\gamma^\mu \overleftrightarrow{D}(z)^\nu \psi(z) + (\mu \leftrightarrow \nu)] - g^{\mu\nu}\mathcal{L}_q$$

where $\mathcal{L}_q$ is the quark part of $\mathcal{L}_{qG}$.

Since $g^{++} = 0$

$$t^{++}_{q,\,bel}(0) = \frac{i}{2}\{\bar{\psi}(0)\,\gamma^{+}\overleftrightarrow{D}^{+}\,\psi(0)\}$$

so that

$$\int_0^1 dx\, x\,[\,q(x) + \bar{q}(x)\,] = \frac{1}{2(P^+)^2}\langle\, P\,|\, t^{++}_{q,\,bel}(0)\ |\, P\,\rangle.$$

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Consider the physical interpretation of the LHS in the parton model. The parton model is not synonymous with QCD. It is a picture of QCD in the gauge $A^+ = 0$ and it is in this gauge, and in an infinite momentum frame that x can be interpreted as the momentum fraction carried by a quark in the nucleon.

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Hence the fraction of *longitudinal* momentum carried by the quarks in an infinite momentum frame is given equally well by either the canonical or Belllinfante versions of the energy momentum tensor density.

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But this J_z is the Bellinfante version! Does it mean that the RHS is not our interpretation of the angular momentum?

Need to know connection between matrix elements of $t^{\mu\nu}$ and matrix elements of J. To 1st order in $\Delta = P' - P$

$$\begin{aligned}\langle P', S' \,|\, t^{\mu\nu}_{q,\,bel}(0) \,|\, P, S \rangle &= [\bar{u}'\gamma^{\mu}u\,\bar{P}^{\nu} + (\mu \leftrightarrow \nu)]\mathbb{D}_{q,\,bel}(\Delta^2)/2 \\ &- \left[\frac{i\Delta_\rho}{2M}\,\bar{u}'\sigma^{\mu\rho}u\,\bar{P}^{\nu} + (\mu \leftrightarrow \nu)\right][\mathbb{D}_{q,\,bel}(\Delta^2)/2 - \mathbb{S}_{q,\,bel}(\Delta^2)] \\ &+ \frac{\bar{u}'u}{2M}\left[M^2\mathbb{R}_{q,\,bel}(\Delta^2)g^{\mu\nu}\right]\end{aligned}$$

where

$$u \equiv u(P, S) \qquad u' \equiv u(P', S').$$

$$\langle\, \psi_{\boldsymbol{p},\boldsymbol{s}} \,|\, M^{ij}_{bel} \,|\, \psi_{\boldsymbol{p},\boldsymbol{s}} \,\rangle \;=\; \frac{1}{M}\Big\{\frac{\mathbb{D}_{bel}}{2(p_0+M)}\,(p^j\,\epsilon^{0i\alpha\beta} - p^i\,\epsilon^{0j\alpha\beta}) + \mathbb{S}_{bel}\,\epsilon^{ij\alpha\beta}\Big\}\,S_\alpha\,p_\beta \qquad (4)$$

The $\mathbb{D}_{bel}$ term vanishes in the M^{12}_{bel} if $\boldsymbol{p}$ is along OZ.

Thus, for a longitudinally polarized nucleon moving at high speed in the Z direction $\mathbb{S}_{bel}$ measures the Z-component of $\boldsymbol{J}$.

So, Ji sum rule becomes

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Parton model interpretation: choose gauge $A^{+} = 0$. Recall $t^{++}_{q,\,can}(0) = t^{++}_{q,\,bel}(0)$, so that

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$$\mathbb{S}_{q,\,bel} = \mathbb{S}_{q,\,can}.$$

Thus $J_{bel,z}$(quarks)=$J_{can,z}$(quarks) and Ji sum rule is OK.

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It shouldn't: can show projection of the spin terms onto the direction of the photon's (or gluon's) momentum i.e. the photon (and gluon) helicity, is gauge invariant and it is this quantity which can be measured in deep inelastic scattering on atoms or nucleons respectively.

Summary

- There is no need to insist that the operators appearing in expressions for the momentum and angular momentum of the constituents of an interacting system should be gauge invariant, provided that the *physical matrix elements* of these operators are gauge invariant.

- We suggest that *the minimal requirement for identifying an operator P_E^j with the momentum carried by E* , is to demand that *at equal times*

$$i[P_E^j(t)\,,\phi^E(t,\boldsymbol{x})] = \partial^j\phi^E(t,\boldsymbol{x}).$$

Analogously, *for an angular momentum operator M_E^{ij} ($J^i = \epsilon^{ijk}M^{jk}$) we suggest* that at equal times

$$i[M_E^{ij}(t)\,,\phi_r^E(t,\boldsymbol{x})] = (x^i\partial^j - x^j\partial^i)\phi_r^E(t,\boldsymbol{x}) + (\Sigma^{ij})_r{}^s\phi_s^E(t,\boldsymbol{x})$$

where r and s are spinor or Lorentz labels and $(\Sigma^{ij})_r{}^s$ is the relevant spin operator.

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- The expressions given by Chen et al and Wakamatsu for the momentum and angular momentum operators of quarks and gluons are somewhat arbitrary and do not satisfy the fundamental requirement that these operators should generate the relevant infinitesmal symmetry transformations.

- Demanding that these conditions be satisfied leads to the conclusion that the canonical expressions for the momentum and angular momentum operators are the correct and physically meaningful ones.

- It is then an inescapable fact that the photon and gluon angular momentum operators cannot, in general, be split in a gauge-invariant way into a spin and orbital part. However, the projection of the photon and gluon spin onto their direction of motion i.e. their helicity, is gauge-invariant and is measured in deep inelastic scattering on atoms or nucleons respectively.

- Although Ji's expressions for the quark and gluon angular momenta are the Bellinfante versions, it turns out that the expectation value of the Bellinfante operator $J_{z,\,bel}$(quark) used by Ji for the *longitudinal* component of the quark angular momentum, which has the nice property that it can be measured in deeply-virtual Compton scattering reactions, does indeed represent the Z-component of the angular momentum carried by the quarks in a nucleon moving in the Z direction.